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ABSTRACT

This Comprehensive School Mathematics Program (CSMP) guide is divided into three major parts. The first, General Introduction, contains: (1) CSMP Viewpoint; (2) Program Description; (3) Guidelines for Evaluating the Program; (4) Commonly Asked Questions about the CSMP Curriculum and Suggested Answers, and (5) A Note to the Parents of CSMP Students. Part 2, The World of Numbers, opens with a suggested chronological order of lessons and an introduction that covers information about standard algorithms and notes on calculator use. Following this are chapters on: (1) Detective Stories and Composition; (2) Rational Numbers; (3) Hand-Calculator Puzzles; (4) Combinatorics; (5) Multiples and Divisors; (6) Division, and (7) Cartesian Graphs. The section concludes with an appendix on standard configurations and minicomputer plays. The final major part, Workbooks, presents a Workbook Progress Chart, and has sections that present answers to many of the various workbook problems. (MP)

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Comprehensive School Mathematics Program
CEMREL, Inc.

Experimental Version

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# CSMP Mathematics for the Intermediate Grades

## Part V

## Teacher's Guide

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General Introduction
The World of Numbers
Workbooks

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Comprehensive School Mathematics Program

# Mathematics for the Intermediate Grades
## Part V

CEMREL, Inc.

# PREFACE

CSMP is firmly committed to the proposition that the study of mathematics should be an interesting, challenging, intellectual and aesthetic experience for all children. The teachers who will teach its curriculum should be similarly committed to that proposition. The CSMP curriculum expects to truly engage children in the study of mathematics. It expects to call upon their curiosity, their intuition, their powers of reasoning, their imagination, their perseverance, and their organizational skills. We believe that the study of mathematics not only calls upon these faculties of children, but will help to develop them. But the curriculum cannot do the job by itself -- there must be teachers who share the basic intellectual values built into the curriculum.

There is an almost infinite variety of uses to which mathematics can be and has been put. Certainly, a reasonable mastery of arithmetic skills is part of the mathematical equipment of a competent, modern adult. Our job as educators is to provide children with a set of mathematical experiences far broader than the acquisition of simple arithmetic skills. We believe it is important to expose children to a wide variety of ways in which mathematics can help them to formulate and think about problems. We believe further that the arithmetic skills themselves can be much more successfully taught in the context of a rich and challenging mathematical environment. It is especially important to prepare children with a variety of mathematical activities in view of the fact that no one can possibly know what career and life-style an elementary school child will ultimately want to choose.

It is CSMP's conviction that mathematics is a unified whole and should be learned as such. The organizational basis of the CSMP curriculum is the spiral approach. Thus, the program is sequenced so that students come into contact with each area of content continuously throughout the program. This approach consciously precludes atomizing the content and mastering each bit before continuing to the next. Rather, students encounter repeatedly the ideas, building interlocking experiences of ever increasing sophistication.

CSMP believes that no single method of classroom management suffices to meet the needs of every student. Hence, the program is constructed to allow numerous opportunities for whole class participation, small group participation, and independent individual study.

The CSMP materials are constructed to maximize teacher-student and student-student interaction. Children begin immediately to investigate mathematical concepts in concrete situations appealing and meaningful to them at their level of experience.

Individualization at this level is achieved through graded worksheets and workbooks with problems at varying levels of difficulty all in one booklet. Worksheets and workbooks are integrated into the basic sequence of lessons. Due to the spiral nature of the curriculum, each student can engage in a workbook for a period of two (possibly more) weeks without losing touch with the progress of the class and, at the same time, working his or her way to the limits of his or her capacity.

Unlike most modern programs, the content is not presented as an artificial structure external to the experience of children and into which they are thrust but rather as an extension of experiences children have encountered in their development, both at the real life and the fantasy levels. Children are led through sequences of problem solving experiences presented in game-like and other story settings. Nonverbal languages give children immediate access to the mathematical ideas and methods necessary not only for solving these problems but also for continually expanding the students' understanding of the mathematical concepts themselves. These languages include:

1. The language of strings. This language mimics the basic method of organizing and collecting data related to problems at hand. That is, it deals with the fundamentally useful and important mathematical notion of sets.

2. The language of arrows. This graphical language models the fundamental process involved in comparing and analyzing sets and operations on them.

That is, it deals with the useful and important mathematical notions of relations and functions including, but not restricted to, permutations of sets, ordering relations and the numerical processes of adding to, subtracting from, multiplying by, and sharing equally among.

3. The language of the Papy Minicomputer. The Papy Minicomputer, an extremely effective abacus invented by Georges Papy, models the positional structure of our system of numeration and lends itself to very sophisticated algorithmic processes, including the standard algorithms for the basic numerical operations.

Through these languages the children are able to apply the basic mathematical ideas effectively and creatively to explore the content directly without wading through the cumbersome, sterile terminology and notation which, in most other programs, pervade the early stages of the curriculum and stand as barriers between the child and the content.

The content is learned in an atmosphere of constant connection with applications, from simple story situations in the early grades to challenging nontrivial simulations of real world problems in the upper grades. The emphasis is at all times on a two level approach to learning: understanding the content and its applications, and equally important, developing the techniques and processes of learning the content. It is the latter form of knowledge which gives the power to apply the former.

# NOTE ON AUTHORSHIP

"Teacher's Guide to CSMP Mathematics for the Intermediate Grades, Part V" is the product of the concerted efforts of a number of people over several years' time. This edition was produced during the 1977-78 academic year by the following members of the CSMP staff and with substantial contributions from the staff associates.

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The lessons in this guide were developed and tried out with two sixth grade classes at the Brittany Middle School, University City, Missouri. The students, representing a full range of abilities, had been introduced to the CSMP curriculum at the beginning of fourth grade in a one semester program using "CSMP Mathematics for the Intermediate Grades, Entry Module I". The entry program prepared the students to begin the first part of the CSMP Intermediate Grades Program for veteran students during the second semester of the fourth grade.

We are extremely grateful to Mr. Jack Fitzgerald, Mathematics department head at Brittany School, for his cooperation in arranging for CSMP's teaching of the two classes for development purposes. Our greatest motivation for this work was the enthusiasm and steady progress of the students.

# GENERAL INTRODUCTION

Comprehensive School Mathematics Program

# Mathematics for the Intermediate Grades
## Part V

General Introduction

CEMREL, Inc.

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## 1. The CSMP Viewpoint

The CSMP curriculum is based on the belief that mathematics belongs to that part of the school curriculum which deals with ideas rather than that part which stresses techniques. This is not to say that there are no techniques involved in the learning of mathematics. There are, just as there are in the learning of English. But these techniques, like spelling and grammar in the learning of English, are not ends in themselves; they are important in so far as they help children to understand and use some area of mathematics.

What are the objects of mathematical study? For us, they include numbers of various kinds; operations on and relationships between numbers; geometrical figures and their properties; relations and functions (e.g., the 2× function, the "less than" relation, etc.) and operations on them.

Among all objects of mathematical study, from the simplest to the most complex, there is one common property: if we separate these objects of study from the real life situations that they model, they are abstract; that is, they are things of the mind. For this reason, mathematics is essentially an intellectual activity. For example, the number 2 represents the phenomenon of twoness, a quality which is present in an endless variety of contexts in life. We can picture the number 2 by two dots or by a stick two centimeters long, and these representations are certainly useful in suggesting to our minds what may or may not be true about the number 2. But ultimately the number 2 is accessible to us only through our minds. In the same sense, a geometrical line is neither a pencil line nor a stretched string nor a ray of light, but is an abstraction which may be pictured in many ways.

This is surely an adult point of view. But it is one that should be understood and taken into consideration by those who develop mathematical materials for children and those who teach the materials to children.

Since mathematical objects are ideas, one of the ways we explore the properties of and the relationships among such objects is to think about them. It is for this reason that growth in the ability to reason plays such an important role in the study of mathematics. We strongly argue, however, that in mathematics the development of the art of reasoning goes hand in hand with the growth of imagination, ingenuity, and intuition. At the elementary level, mathematical objects, their properties, and their interrelationships must be thought about and discussed in various suggestive and interesting situations.

Seen in this perspective, elementary arithmetic then takes the form of "adventures in the world of numbers". Individual numbers such as 0, 1, 2, 100, $\frac{1}{5}$, $^{-}7$, $\frac{^{-}4}{9}$, 3.72, $^{-}1.006$, and so on can assume definite personalities of their own in this world. The teaching of arithmetic shifts its emphasis from an obstacle race for mastery of the standard algorithms to the stimulation of exploring the world of numbers. Skills such as balancing a checkbook have their place in real life but are of absolutely no interest to a fifth or sixth grader. It is unrealistic to view such activities as goals of an elementary school mathematics education. Far more genuinely rewarding are the stretching of the powers of imagination and the challenging of the mind. Facility grows naturally, along with knowledge and understanding. Necessary skills become byproducts of such an educational experience.

Such a curriculum requires a special kind of pedagogy. The pedagogy employed should include teaching strategies which encourage creativity and allow freedom to explore but still foster sound intellectual habits. Attitudes toward student achievement that allow for intellectual and emotional growth and respect the individuality of each student are essential to the successful attainment of the goals of this type of curriculum.

The pedagogy which CSMP develops in this spirit can be described as a "pedagogy of situations".[*] The introductions to the various strands of the CSMP Intermediate Grades Program and those to the chapters within the strands refer to many examples which illustrate aspects of this pedagogy. All these examples are taken from the lessons themselves, and a prime source of

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[*] This phrase is due to the mathematician and psychologist, Caleb Gattegno.

"pedagogical situations" is the lessons of the program. At the risk of over-simplification, a pedagogy of situations may be described as one which is based on the belief that learning occurs in reaction to the experience of confronting a situation (real, simulated, or imagined) that is rich in consequences, is worthy of confrontation, and has genuine intellectual content.

Children can react to a situation that intrigues them. They may grapple with problems of symbolization and the relation of a symbol to the thing it represents. Arguments about the consequences of the given information may be close to those which are the stuff of mathematics. Although adults may protest that some of the situations in the lessons are not "real", they are real to children, who can often involve themselves in and feel at home with "pretend" situations more readily than with many "real" situations. On the basis of several years' experience in a wide variety of classes, we can say with confidence that a pedagogy of situations:

- involves children at a personal level (they are interested in the situations presented, and sometimes want to dramatize or devise related situations);
- presents an intellectual challenge that is accessible to a broad range of abilities (even the slowest students understand the challenge and can learn something from their experience in confronting a situation; some may even be able to realize the full learning potential of that challenge);
- provides opportunities for creativity (some children are motivated by well chosen situations to generalize from their experiences or to make conjectures to be tested in similar situations);
- supports experiences that have mathematical content.

The CSMP curriculum fulfills what we consider to be the fundamental role of mathematics in the elementary school by presenting a large number of varied, yet interrelated situations that provide the experiences out of which mathematics grows. Two questions immediately arise. How do students learn necessary "number facts" via a pedagogy of situations? And how can we "grade the achievement" of a child as he or she reacts to a situation?

Believe it or not, the "number facts" are among the things which students learn almost inadvertently under a pedagogy of situations. Numbers become personalities; their interrelations are the subject of many stories in which numbers talk to each other and react to each other. In confronting some of these situations, students make various calculations out of necessity. Calculation which serves some higher, more motivating purpose presents nowhere near as many difficulties as does calculation for its own sake. Further, more specific remarks concerning calculation occur scattered through the latter part of this introduction and in the introductions to chapters of the World of Numbers strand.

As for "grading of achievement", CSMP takes the position that learning often stops when the purpose of a lesson or sequence of lessons is the mere mastery of skills. Every child should learn something from each situation; otherwise, it is not a fruitful situation and should be revised or eliminated. But to measure what or how much has been learned in any one situation is probably not possible. Learning is not a linear process. It occurs in quantum jumps that take place at times when you least expect them. It is possible to monitor the progress of students, and you have your own ways of doing this, but who is to say at what precise time everyone in a class is supposed to have gained a particular skill or concept from his or her experiences? We comment further on this matter in the next section of this General Introduction.

A more substantive question is: How can we present situations rich in mathematical content before children have the verbal equipment and the computational skills to deal with them? It is presupposed by many people that children must be able to read and to calculate before they can tackle any mathematical situation other than trivial ones. Yet if one analyzes the types of thinking and the few basic concepts underlying elementary school mathematics, one finds that there is nothing esoteric or even difficult about these thinking modes or these concepts. What has been difficult for students in the past is the language in which these modes of thinking and these concepts are traditionally couched. For some reason, in our natural languages these modes of thinking are difficult to communicate precisely. That is why mathematicians devise languages of their own, languages that minimize the ambiguities of natural languages. Part of the problem in the past has been that elementary school children have found it difficult to deal with such tricky uses of language at a time when they do not yet have a firm command of their natural language.

On the other hand, since the underlying ideas and modes of thinking are inherently simple, there is no reason why very young children should not have the pleasure of such thinking and of exploring mathematical ideas. All that is needed are languages with which to express such ideas and to communicate the thinking that is generated by the ideas. These languages must be very simple, essentially nonverbal, and precise.

CSMP uses three such languages that fit the criteria and that have been highly successful in allowing students to do the kind of thinking that we described above. We call them the language of strings, the language of arrows, and the language of the Papy Minicomputer.

The first two are picture languages: brightly colored strings and dots enable precise descriptions of and the accurate recording of thinking about classifications of objects; colored arrows between pairs of dots stimulate thinking about relations between objects. For detailed descriptions and discussions of these two languages you are referred to the introduction to the Languages of Strings and Arrows strand.

The third language is based on a simple abacus called the "Papy Minicomputer". Using this language, one may represent all decimal numbers (positive or negative) in such a way as to encourage creative thinking about the nature and the properties of those numbers. With the support provided by the Minicomputer, the children, even at an early age, have the opportunity to develop excellent mental arithmetic skills and to gain sophisticated insights into the world of numbers. The ability to perform mechanical computation is one of the incidental by-products of such an approach to work with numbers.

## 2. Program Description

### a) Description of Materials

The CSMP Intermediate Grades curriculum is designed as a program for students in grades four, five, and six. There are six parts to the curriculum, each corresponding to one semester's work. This is Part V of the six.

In addition, there are two Entry Modules, one each for grades four and five. As the name implies, each Entry Module is designed to facilitate transition into the CSMP Elementary School Curriculum.

The materials of Part V of CSMP Mathematics for the Intermediate Grades require some explanation. This collection of two volumes (six books) which are directed specifically to you, the teacher, is known as the "Teacher's Guide to CSMP Mathematics for the Intermediate Grades, Part V". (By way of abbreviation, in what follows we shall refer to this collection simply as the "Teacher's Guide".) With the exception of this book, "General Introduction". each book of the Teacher's Guide addresses itself to a single area of the curriculum, and each of these areas is known as a "strand". The four content strands are:

- the World of Numbers - (N);
- the Languages of Strings and Arrows - (L);
- Geometry and Measurement - (G);
- Probability and Statistics - (P).

The six books comprising the "Teacher's Guide" are bound into two volumes, with a separation page inserted between the various books of a given volume.

This volume contains both this General Introduction as well as the World of Numbers strand. The other volume contains the Languages of Strings and Arrows strand, the Geometry and Measurement strand, the Probability and Statistics strand, and the Workbook strand. Each of these six books has its own table of contents and is paginated individually.

Each strand has the following arrangement of material.

- A table of contents. This enables you to locate whatever you need in that particular strand.
- A suggested chronological order of the lessons in that strand. This provides any easy, quick reference uncluttered by extraneous information. It shows you in which order the lessons described in the strand will be taught if you follow the suggested schedule.
- An introduction to the strand. This explains why certain material finds its place in that strand, what the overall goals of study are, and how this material fits in with that which is described in other strands.
- Suggestions for the planning and conducting of lessons in that strand. These include lesson descriptions, suggestions on the use of worksheets and manipulative materials, general remarks relating to certain sequences of lessons, etc.

The suggested lessons are grouped in chapters, all the lessons described in the same chapter having some particular feature(s) in common. Because of this commonality of purpose, it is possible to precede each chapter by an introduction which indicates where the lessons in that chapter are going and, on occasion, gives a few more details concerning the mathematical underpinnings of the lessons described in the chapter than are contained in the lesson descriptions themselves.

Many of the suggested lessons require student worksheets for individual work.

Classroom sets of these worksheets are printed and packaged separately from this guide. However, an answer key for each of the worksheets is included in the guide. Worksheets are numbered consecutively within each strand and their answer keys can be found at the end of the lessons for which the worksheets are needed. For example, an answer key for Worksheet N35 is at the end of Section 4.1 of the World of Numbers strand since it is suggested that this worksheet be used with the lesson described in that section.

In order to accommodate the various ability levels in a given class, the worksheets generally are written at several levels of difficulty denoted by stars. "*" represents an easy worksheet, "**" represents an average level worksheet, and "***" and "****" represent difficult level worksheets. All students should begin with the "*" level worksheets for a given lesson or sequence of lessons and proceed as far as possible through the other levels. In classes containing the usual range of abilities, most of the students should succeed with the * worksheets; the ** worksheets should be appropriate for about two-thirds of the class; and approximately one-third of the students will find the *** and **** worksheets challenging and enjoyable to do. No one should ever be required to attempt all the worksheets provided for a given lesson. Each student should progress as far as his or her abilities allow.

There are various other items which are (or could be) used in the program. Among them are the following: various manipulative materials such as translators, rulers, compasses, grid board, and so on (these are described in more detail in the lessons in which they are used); several workbooks, storybooks, and story-workbooks (these are described in more detail in Sections 2 (b) and (e) of this General Introduction, and in the introduction to the Workbook strand); a two-volume book entitled "Math Play Therapy", which includes a collection of suggested small group activities that could be instrumental in stimulating an open and positive attitude toward mathematics on the part of those of your students who find the subject difficult. See the introductions to both volumes of "Math Play Therapy" for suggestions as to its use in the program.

### b) Role and Use of Workbooks

It is suggested that approximately one day each week be scheduled for students to do individual work in designated workbooks. The workbooks are designed to review ideas encountered in the content strands and to apply these ideas to new situations posed as stories. Workbooks are written with problems of varying levels of difficulty in order to accommodate differences in students' abilities and interests.

More detailed descriptions of the workbooks, their purposes, and answer keys are given in the Workbook strand. There you will also find suggestions for keeping records of student progress.

### c) The Spiral Development

The learning process occurs through many interrelated experiences. But no single experience, particularly a 40 to 50 minute mathematics lesson, is an end in itself. The approach in the CSMP curriculum keeps this in mind. It is neither intended nor expected that every student will meet the full challenge of a particular situation in any given lesson. Furthermore, we find that it is most effective to vary the situations and strands from day to day rather than to continue with one type of situation or piece of content until so-called mastery has occurred. Thus, we suggest that the lessons be scheduled so that a different topic is studied each day of the week. In a typical week, for example, your class might work with numerical games one day, probability the next day, workbooks the third day, numerical relations the fourth day, and geometry the fifth day.

The spiral development, in which a student experiences each of several ideas a little at a time and then proceeds through increasing levels of sophistication as the situations become more challenging, reflects the fact that learning is not a linear process. Not only do different children learn at different times and at different rates, but even on an individual level learning can occur in quantum leaps that take place at times when you least expect them. The spiral development gives each student a new chance to catch on to an idea at each turn of the spiral.

Of course, acceptance of the CSMP pedagogy requires a certain act of faith on your part. If you are accustomed to staying with a given topic until everybody (or nearly everybody) in the class knows it "cold", then you may be initially very reluctant to leave a lesson at the end of the time suggested, believing that several students "haven't got it yet". By keeping the spiral idea in mind, you should become more comfortable with these time schedules. Although it may appear to you that little learning is taking place in a sequence of lessons, you may be surprised that suddenly even your slowest students are understanding the ideas when they encounter them later in the spiral. Whereas with a mastery approach, presented in a purely individualized mode, slow students often spend most of the school year trying to "master" low level skills (with generally poor results), the spiral approach gives these students the opportunity to benefit from the richness of the situations presented throughout the curriculum. In this manner students are allowed to be involved in interesting and enjoyable situations which are modeled by important mathematical notions. They are not deprived of a variety of experiences just because they haven't mastered every bit and piece along the way. Therefore, in the long run, their interest in, understanding of, and even facility with mathematics has a much better chance to develop.

### d) Evaluation

On the basis of the preceding discussion, it should come as no surprise, as stated earlier, that CSMP takes the position that learning often stops when the purpose of a lesson or sequence of lessons is the mere mastery of skills. Furthermore, we do not believe it is desirable or reasonable to attempt to assess student mastery of skills from day to day or even from week to week. As we said before, learning has a habit of occurring when you least expect it, so how does one determine when everyone in a class should have mastered a specific skill or attained understanding of a particular concept as a result of his or her experiences? We know of no method for measuring what or how much has been learned from any one situation except in the most trivial cases. On the other hand, teachers, parents, and even the students themselves can benefit from some continuous monitoring process that records involvement in activities and assesses progress through the program. Indeed, it is possible to monitor your students' progress without contradicting the statements made above; all of you have your own ways of doing this. Such a process is suggested in the introduction to the Workbook strand. Of course, you should also continue to use any monitoring devices with which you feel comfortable.

As far as traditional computational skills are concerned, the standardized test data collected during the extended pilot test stage of the development of this curriculum indicate that CSMP classes generally do better than average on the items which test "basic" skills.

About 75% of the lessons for the CSMP Intermediate Grades curriculum involve numerical concepts. The situations involve not only positive rational numbers (as is generally the case at this level), but introduce the students to many other citizens of the world of numbers at as early a stage as possible. You should not infer from this that your students are expected to have mastered the arithmetic of the whole numbers, of the integers, of the decimal numbers, and of the rational numbers before they are ready for Part VI of the CSMP Intermediate Grades curriculum. On the contrary, it means that the students will become more and more friendly and at ease with the various types of numbers and will develop their own methods for calculating with them long before some standard mechanical algorithms are introduced.

It is essential that the creative freedom of the student not be impaired by a premature emphasis on the standard algorithms for adding, subtracting, multiplying, and dividing. We believe that many of the present computational difficulties experienced by elementary and junior high school students were caused by their spending an inordinate amount of time trying to master the algorithms too early in their mathematical education. This shuts the students off from other experiences necessary for their mathematical growth. Students need plenty of time to explore the world of numbers. Mastering an algorithm should be the last step of this exploration, a sort of natural synthesis of previous experience. Furthermore, as electronic hand-calculators become available to nearly everyone, algorithmic mastery as a prerequisite to other numerical experiences becomes less important.

We strongly urge you to follow the suggested schedule of lessons, teaching the algorithms as they come, but not giving them more emphasis than they deserve. Skill in mental arithmetic and estimation, however, is extremely important in our opinion and should be promoted at every opportunity.

While CSMP disagrees with the philosophy underlying standardized testing, we realize that students as well as teachers are under extreme pressure to perform in certain ways with regard to such tests. If your students or their parents express concern about the apparent lack of emphasis on mastery of standard algorithms (such as long division by three or four digit numbers), they should be reassured that CSMP students have a definite advantage when it comes to applying mathematics to novel situations, thinking strategically, organizing data or information, and understanding methods of mathematics. Because CSMP feels it is wrong for such tests to dictate emphasis on the mastery of standard algorithms and that drill for algorithmic mastery for its own sake defeats CSMP's purposes, you will find that even after algorithms have been introduced into the CSMP curriculum the role they play will be as tools for doing other interesting things. Therefore, if you feel pressured by standardized tests or by other measures to put more emphasis on drill activities, then we suggest you use any supplementary materials that have proved successful in the past to provide this extra practice. We leave scheduling choices for these kinds of activities to your judgment.

### e) Role and Use of Storybooks and Story-Workbooks

Among the instructional materials for the CSMP Elementary School curriculum is a collection of mathematical storybooks, STORIES BY FREDERIQUE. This collection consists of twenty-four stories for young people, presented in colorful newsprint booklets. They are intended to appeal to youngsters in the 5-14 age bracket. The stories are completely independent of each other and need little or no external explanation. They may be used for individual or collective reading, at home or at school.

Each of the stories is portrayed by the mathematical languages of dots, strings, and arrows which reinforce the ideas of the story. Three of the stories (TWO BY TWO, THE WEIRD STORY OF 24, and SUMMER SCHOOL IN THE OLD DAYS) also use the language of the Minicomputer. The amazing versatility of these nonverbal languages is fully exploited.

Young readers can become personally involved in these stories; there are questions to answer, pictures to relate to the stories, and new adventures to invent for the characters in the stories. The reader discovers that emotions and aesthetic senses can be involved in the world of mathematics just as in the world of people.

Twelve of the stories are intended for very young children (5 to 8 years of age). Here the focus is on the anecdotal aspect of the story. One of the goals of these twelve books is to familiarize children with several nonverbal mathematical languages and to leave them with the impression that these languages are as natural and useful as the usual language.

Nine of the remaining stories concern numbers and their relationships with the world. These particular stories are about a boy and some of his friends who just happen to be numbers. The personification of numbers introduces the "games" of mathematics into one of the most important domains of childhood -- the world of fantasy. Free from the usual constraints so often placed upon them, the numbers invent marvelous games and dance to the tunes of vividly colored arrows. These games are never frivolous. There are rules to be followed and as a result, orderly patterns emerge as the numbers play or dance.

Of the remaining three books, two are detective stories introducing some interesting mathematical problems, and the other one emphasizes the various and different aspects of a simple straight line.

The titles of the books in the collection and the intended age range for each story are:

Ages 5 to 8

THE PLAYFUL NUMBERS
81 ROSES
I AM A VERY HAPPY BOY
ONE OUT OF SEVEN
THE HAPPY PUPPET
TWO BY TWO
THE LITTLE DREAMER
THE LITTLE DONKEY
THE MAGIC BOX
THE BABY IS BORN
THE WEIRD STORY OF 24
THE OLD SHOEMAKER

Ages 8 to 12

WHERE'S MY NOSE?
SINGING FRIENDS
DANCING FRIENDS
THE LIVING LINES
I AM NOT MY NAME
NABU WINS AN AWARD
THE SQUARE TRAP
SUMMER SCHOOL IN THE OLD DAYS

Ages 10 to 14

THE HIDDEN TREASURE
A VERY STRANGE NEIGHBORHOOD
ELECTION IN THE NUMBER WORLD
A VALENTINE MYSTERY

In the classroom set of materials for Part V of the Intermediate Grades Program, two copies each of the following storybooks are provided: TWO BY TWO, WHERE'S MY NOSE?, SINGING FRIENDS, DANCING FRIENDS, THE LIVING LINES, I AM NOT MY NAME, NABU WINS AN AWARD, and THE SQUARE TRAP. They should be put in the class library for students to read whenever they like. These books are in the third grade program or in previous parts of the Intermediate Grades Program; students may enjoy reading them from time to time. If Part V is being used in the second semester, following Part IV in the first semester, these storybooks are not included with Part V materials since they were provided with Part IV materials.

One of the storybooks, ELECTION IN THE NUMBER WORLD, is the basis for three lessons in the Workbook strand. The classroom set of materials for

Part V contains fifteen copies of this storybook; however, you may wish to order additional copies so that every student may have a copy.

Besides the collection, STORIES BY FREDERIQUE, another collection, THE CSMP LIBRARY, has been developed as part of the instructional materials for the CSMP Elementary School curriculum. This is a series of story-workbooks which combines the spirit of the storybooks and the workbooks into a single integrated experience for the children, engaging them in reading about and doing mathematics in stimulating, challenging story situations. The current list of story-workbooks in the collection is:

Ages 8 to 11

SUMMER SCHOOL - 0'S DISCOVERY
ROLLERSKATING 37
TO PICTURE
A SHORT STORY ABOUT 2 FRIENDS
A SHORT STORY ABOUT 3 FRIENDS
THE ISLAND OF TAM - TAM
THE TALKATIVE NUMBERS

Ages 9 to 14

SUMMER CAMP
HALLOWEEN PUZZLES
1,000'S DREAM
A STRANGE COUNTRY
NOT TOO CLOSE
WHO ARE WE?
CLINTON STREET
A BOOK ABOUT ME
SEVEN SECRET NUMBERS
SHUNDA'S NEWSSTAND

### f) Role and Use of Hand-Calculators

The electronic hand-calculator, if used creatively, can have a remarkably beneficial effect on the students' interest in and understanding of arithmetic, and its use can be a contributing factor in the improvement of their capabilities in that area. The reasons for this are many and varied, but among them are certainly the following:

- By removing the burden of mechanical computation from the children's shoulders, the calculator frees them to concentrate on the thinking that is a necessary part of the solution to any meaningful problem.

- The instant feedback provided by the calculator enables the children to indulge in making and testing conjectures, an area into which they would not otherwise venture because the quantity of computation involved would be too forbidding.

This pedagogical tool is particularly useful in the introduction of new numerical notions such as negative integers, decimal numbers, multiples of a given whole number, and so on, because through suitably chosen, rich situations these concepts can be made to arise in a natural way.

A classroom set of battery powered hand-calculators is recommended for Part V. Many different kinds of hand-calculators are available from CSMP as a distributor for National Semi-Conductor and Texas Instruments. Any of the products of these two companies may be purchased at discounted prices.

If it is not possible to supply all students with hand-calculators, encourage the students to bring calculators from home. The hand-calculators that are available from CSMP have algebraic logic and function keys which are permanently in a constant mode. You should check to be sure that calculators that you supply or that are brought in by students also have these characteristics.

- Algebraic logic. This means that the calculator responds to instructions given in the order in which they are usually written. To check that this is the case, press $\boxed{2}\ \boxed{+}\ \boxed{3}\ \boxed{=}$. If you have "5" on the display, then all is well.
- Constant mode. Some calculators have a switch so that the calculator can be set either in a constant mode or in a chain mode. If your calculator lacks a switch, it is quite likely that the calculator has an automatic constant. For calculators with an automatic constant, the first number entered during multiplication is usually the constant, while the second number entered during the other three operations (addition, subtraction, division) is usually the constant.

To check your calculator for the constant mode operation of $\boxed{+}$ is a simple procedure. Press $\boxed{0}\ \boxed{+}\ \boxed{2}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ and so on. If the number on the display is increasing by two at each depression of $\boxed{=}$, than $\boxed{+}$ is operating in a constant mode. Check the constant mode operation of $\boxed{-}$ in a similar fashion.

To check the constant mode operation of $\boxed{\times}$ (where the first number entered will be the constant) press $\boxed{2}\ \boxed{\times}\ \boxed{1}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ and so on. If the number on the display doubles at each depression of $\boxed{=}$, then $\boxed{\times}$ is operating in a constant mode.

While it would be possible to use calculators without these characteristics in the lessons, the adjustments that would be required in the instructions make the calculators unsuitable.

The hand-calculator relations $\boxed{+}\ \boxed{n}\ \boxed{=}\ \ldots$ and $\boxed{+}\ \boxed{=}\ \ldots$ are explored quite extensively in several lessons. Even if your calculators have algebraic logic and the constant mode, it may be necessary for you to check to see if your calculators operate in the way that is described in the lessons.

If you find that your calculators do not operate in this manner, then try letting

$$\boxed{+}\ \boxed{n}\ \boxed{=}\ \ldots \qquad \text{mean}$$

$$\boxed{+}\ \boxed{n}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \ldots$$

and

$$\boxed{+}\ \boxed{=}\ \ldots \qquad \text{mean}$$

$$\boxed{n}\ \boxed{+}\ \boxed{0}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \boxed{+}\ \boxed{n}\ \boxed{=}\ \ldots$$

where $n$ is some starting number.

Furthermore, with some calculators there is a correspondence between a multiple of a number and the number of times $\boxed{=}$ is pressed in the $\boxed{+}$ $\boxed{=}$... relation. For example,

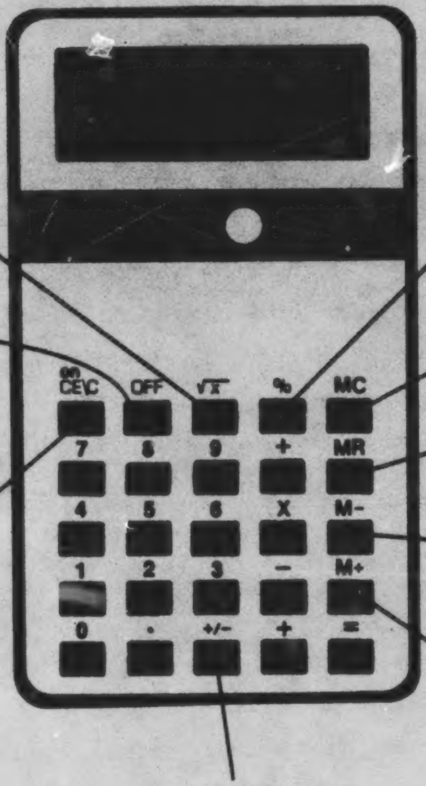
> Starting with "4" on the display and pressing $\boxed{+}$ $\boxed{=}$ yields 4 (= 1 x 4). Pressing $\boxed{=}$ again yields 8 (= 2 x 4); pressing $\boxed{=}$ again yields 12 (= 3 x 4); pressing $\boxed{=}$ again yields 16 (= 4 x 4), pressing $\boxed{=}$ again yields 20 (= 5 x 4); and so on. Notice that when $\boxed{=}$ has been pressed five times, the fifth positive multiple of 4 (i.e., 20) is displayed.

Frequently, calculators that do not behave in the manner described above will differ by one less (i.e., the fifth positive multiple of 4 appears after pressing $\boxed{=}$ four times).

The calculators used in the descriptions of the lessons of Chapter 7, HAND-CALCULATOR PUZZLES, are this latter kind. Therefore, there may be some discrepancy between solutions your students find and the ones presented in those lesson descriptions. However, the differences should be minor and you should be able to adjust the lesson description to fit the type of calculators your students are using.

In Part V you and your class will have several opportunities to explore the use of "special keys" that appear on many hand-calculators (see Chapter 3 of the N strand). The illustration on the following page gives a brief description of some "special" keys that might be on your hand-calculators.

Prior to the presentation of the hand-calculator lessons, you should familiarize yourself with these keys.



Square Root Key — finds a number which, when multiplied by itself, will equal the displayed number

Off Key — clears display, memory, and calculations in progress

On/Clear Entry/ Clear Key — clears an erroneous entry on the display; does not clear memory

Percent Key — allows direct calculation of percent

Memory Clear Key — clears memory; does not clear display

Memory Recall Key — displays contents of memory

Memory Subtract Key — subtracts displayed number from memory

Memory Add Key — adds displayed number to memory

Change Sign Key — changes sign of number displayed; allows entry of negative numbers

## 3. Guidelines for Organizing the Program

The design and content of the lessons in Part V affords you the utmost flexibility in teaching the program successfully. It is a truism to say that your students this year are very different, both individually and as a class, from the students you taught last year, and indeed from other students in any other CSMP class. And your own teaching style and interests are probably very different from those of your colleagues. It is difficult for those of us who develop curricula to guess correctly as to which topics will be very easy for your students and which lessons will require more instructional time. You are the only one who will be able to judge that as you begin using the materials. This program is not meant to be a static sequence of exercises to be followed rigidly; the schedule and design of lessons in this guide should be regarded as suggestions; they simply reflect our own experience teaching the materials. You and your students will make the program dynamic as together you explore the mathematical ideas and follow your own interests and inclinations.

### a) Use of Schedules

If this is the first time you are teaching a CSMP class and you would like specific suggestions for organizing the lessons, we offer some guidelines and a suggested schedule of lessons. Please remember that our suggested sequence is but one of many that could have been assembled. You may wish to follow it at the beginning of the school year and then deviate from it as the needs of your students dictate.

The master schedule included at the end of this section reflects our suggestions for a hypothetical sixth grade class of average ability, and it illustrates some important ideas about organizing the program. You will notice that several different strands are scheduled each week and that even within a strand the topics are interspersed. This was done to carry out the idea of a spiral curriculum. If you organize your own schedule, it is best to follow a similar pattern rather then to spend two or three full weeks, for example, "covering" one particular topic such as composition of relations or decimal numbers. In a usual five day week, two days should be devoted to the World of Numbers strand, one day to the Workbook strand, and the other two days should be shared by the P, G and L strands.

In the schedule, a block represents one full period of approximately 45-50 minutes duration. It may very well be, however, that you and your class will spend more than one period on some lessons. Exercise your own professional judgment in deciding when to stop one particular lesson and when to take it up again. Just be sure that you do not insist that every single child understands every single aspect of a lesson before you go on. It is better to stop a lesson at a point when student interest is high and a problem remains unsolved than to teach so methodically that everyone in the class understands everything, but is completely bored.

Inside each block, the lesson of the day is designated in several ways. First, there is listed a number which indicates its suggested chronological order among the lessons of a strand. Hence, "5N" indicates that the lesson in question in suggested as the fifth lesson in the N (World of Numbers) strand. This identifying number also appears in the upper outside corner of each page in the description of that lesson, thus enabling a lesson to be located simply by flipping through the pages of the strand book. Second, you are given the section number and title of the lesson. Third, you are told on which page of the strand book the description of the lesson in question begins.

Please bear in mind that the model schedule was designed for a hypothetical class with the usual range of abilities. In order to be successful with the program, you should adjust the schedule to the ability, maturity, and interests of your class. For example, you may wish to extend a lesson or a workbook for an additional period beyond those scheduled.

### b) Use of Adjustment Days

From time to time an adjustment day appears in the schedule. Since we have intended that Part V would be the mathematics curriculum for one semester of the Intermediate Grades Program and since a usual school semester is eighteen weeks, the schedule indicates eighteen weeks of lessons. However, it is also usual for there to be days when no mathematics lesson is taught; e.g., parent-teacher conference days, field trip days, snow days, and so on. These adjustment days should account for such non-class days, and should also give you opportunities to do many kinds of things. You could use an adjustment day to review a lesson you feel needs review, or to complete an unfinished lesson, or to extend some interesting ideas from completed lessons. You could also use an adjustment day to give your students time to become more familiar with the manipulative materials in the program, such as the translators or the compass, or with the storybooks. Your students may especially enjoy playing games such as Minicomputer Golf, Minicomputer Tug-of-War, Guess My Rule, the String Game, or the Table Game and an adjustment day could give them another opportunity to play a game. You may wish to use an adjustment day to do some mental arithmetic exercises, or to teach a topic in mathematics you think would be especially interesting or useful to your class.

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# SUGGESTED SCHEDULE OF LESSONS

## 4. Commonly Asked Questions about the CSMP Curriculum - and Suggested Answers.

As you teach the CSMP curriculum (or even as you prepare to teach it) there will surely be some basic questions that come to your mind. They are the kind of questions that not only you as a teacher, but also parents and supervisors, will ask about any curriculum that appears to be different from the usual program.

The following questions and answers are listed as an aid to you in understanding our curriculum and in responding to those who request information about it.

1. Q: FOR WHAT SORT OF STUDENT IS THE CSMP ELEMENTARY CURRICULUM DESIGNED?

   A: It is designed for all children in the schools. Each learning situation is designed to involve all students at their own levels in such a way that even the slowest can succeed in dealing with some aspects of the situation that challenge them. In fact, it has been found that such a "pedagogy of situations" (see Question 2) is particularly effective with the so-called "slow" learners, and recent research suggests it might be effective also with educable mentally handicapped children.

2. Q: BRIEFLY, HOW DOES CSMP DIFFER FROM A TRADITIONAL MATHEMATICS CURRICULUM?

   A: The core content of CSMP is not revolutionary. The program aims to inculcate notions and facts which have always been considered important in mathematics. The major differences are in method. Traditional teaching takes pride in disciplining the student to be submitted to the usual algorithms. CSMP students are involved as well in many other real mathematical activities, with understanding at their level.

For that purpose CSMP uses new teaching aids, the nonverbal languages of strings and arrows and of the Papy Minicomputer, which are themselves both pure and applied mathematics. These mathematical objects (graphs, arrows, etc.) are used as aids rather than as ends in themselves.

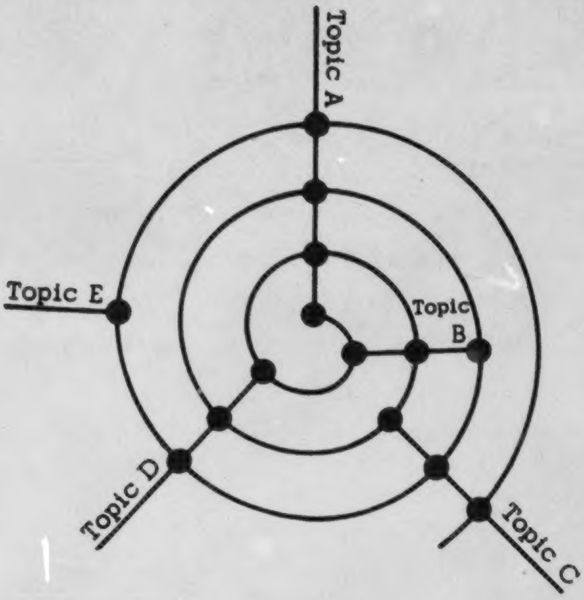
A pedagogy of situations is used to permit children to participate with creative freedom in the learning process. The students are faced with situations and are led by their reactions to important ideas in mathematics. Thereby, students learn to mathematize situations, a process which is of fundamental importance whenever mathematics is applied.

CSMP also makes use of a spiral approach (see Question 3) rather than the mastery approach found in most mathematics curricula.

3. Q: WHAT IS THE SPIRAL APPROACH? HOW DOES IT WORK?

A: The learning process occurs through many interrelated experiences. But no single experience, particularly a 40 to 50 minute mathematics lesson, is an end in itself. The CSMP curriculum was developed with this in mind. It is neither intended nor expected that each child will meet the full challenge of a particular situation in any given lesson. Furthermore, we have found that it is most effective to vary the situations and topics from day to day rather than to continue with one type of situation or piece of content until so-called mastery has occurred. Children at this age do not seem to enjoy staying on the same topic for a very long time but prefer to be involved in a variety of situations. Thus, the lessons are scheduled so that different topics will be studied during each week and similar situations appear again and again at different times and levels in a spiral development.

If a spiral curriculum seems strange at first, consider the following illustration: Imagine an academic subject that could be divided into five subtopics. Then imagine a student's progress being represented by a spiral, starting at the center and traveling outwards in the direction of increasing degrees of maturity and sophistication. In the diagram, an intersection point between the spiral and a topic line indicates a lesson on that topic.



The student is first introduced to Topic A, then to B, and to D, followed by a slightly more sophisticated lesson on A, and so on. Topics C and E are not introduced until a little later in the spiral, while Topic B may be of a more terminal nature and is considered completed after a few lessons.

This spiral development, in which a child experiences each of several ideas a little at a time and then proceeds through increasing levels of sophistication as the situations become more challenging, is wholly consistent with CSMP's pedagogy of situations.

Acceptance of the CSMP pedagogy requires a certain act of faith on your part. If you have been accustomed to staying with a given lesson until everybody (or nearly everybody) in the class knows it

# Master Schedule for the CSMP Intermediate Grades, Part V

| Week | World of Numbers | Languages of Strings and Arrows | Workbooks | World of Numbers | Geometry and Measurement Probability and Statistics |
|---|---|---|---|---|---|
| 1 | 1N<br>1.1 Who is Tob?<br>N p. 3 | 1L<br>1.1 Games with Strings #1<br>L p. 3 | 1W<br>Arcade of Problems #1<br>W p. 2 | 2N<br>2.1 Decimals #1<br>N p. 35 | 1G<br>1.1 Compass Constructions #1<br>G p. 7 |
| 2 | 3N<br>1.2 Who are Flip and Flop?<br>N p. 13 | 1P<br>1.1 Random Number Stories #1<br>P p. 3 | 2W<br>Arcade of Problems #1<br>W p. 2 | 4N<br>3.1 Hand-Calculator Puzzles #1<br>N p. 104 | 2G<br>1.2 Compass Constructions #2<br>G p. 13 |
| 3 | 5N<br>4.1 Counting Relations #1<br>N p. 125 | 2L<br>1.2 Games with Strings #2<br>L p. 8 | ADJUSTMENT DAY | 6N<br>2.2 Multiplication of Rational Numbers<br>N p. 42 | 2P<br>2.1 Marriage by Chance<br>P p. 38 |
| 4 | 7N<br>5.1 The Relation $=_4$<br>N p. 176 | 3L<br>2.1 Necklaces #1<br>L p. 48 | 3W<br>Arcade of Problems #2<br>W p. 12 | 8N<br>1.3 Composition of Functions<br>N p. 20 | 3G<br>1.3 Perpendicularity #1<br>G p. 21 |
| 5 | 9N<br>2.3 Addition of Rational Numbers<br>N p. 53 | 4L<br>2.2 Necklaces #2<br>L p. 62 | 4W<br>Arcade of Problems #2<br>W p. 12 | 10N<br>6.1 A 7-Abacus #1<br>N p. 251 | 4G<br>1.4 Perpendicularity #2<br>G p. 29 |
| 6 | 11N<br>5.2 The Relations $\sqcup$ and $\sqcap$<br>N p. 187 | 5L<br>1.3 Games with Strings #3<br>L p. 16 | ADJUSTMENT DAY | 12N<br>3.2 Hand-Calculator Puzzles #2<br>N p. 110 | 3P<br>2.2 Who's on Bruce's Team?<br>P p. 47 |
| 7 | 13N<br>5.3 Divisibility by 9 #1<br>N p. 195 | 4P<br>3.1 Probability Games #1<br>P p. 92 | 5W<br>Arcade of Problems #3<br>W p. 22 | 14N<br>2.4 Decimals #2<br>N p. 60 | ADJUSTMENT DAY |
| 8 | 15N<br>5.4 Divisibility by 9 #2<br>N p. 203 | 5P<br>3.2 Probability Games #2<br>P p. 110 | 6W<br>Arcade of Problems #3<br>W p. 22 | 16N<br>6.2 A 7-Abacus #2<br>N p. 264 | 5G<br>1.5 Polygons #1<br>G p. 37 |
| 9 | 17N<br>5.5 Exponents and Prime Factorization<br>N p. 210 | 6L<br>3.1 The Table Game #1<br>L p. 77 | 7W<br>Election in the Number World<br>W p. 32 | 18N<br>2.5 Decimals #3<br>N p. 66 | 6G<br>1.6 Polygons #2<br>G p. 40 |

| | | | | | |
|---|---|---|---|---|---|
| 10 | 19N<br>6.3 Division Algorithm #1<br>N p. 276 | 7L<br>1.4 Games with Strings #4<br>L p. 21 | 8W<br>Arcade of Problems #4<br>L p. 66 | 20N<br>4.2 Counting Relations #2<br>N p. 137 | 6P<br>2.3 Time is of Importance<br>P p. 60 |
| 11 | 21N<br>2.6 Decimals #4<br>N p. 73 | 7P<br>1.2 Random Number Stories #2<br>P p. 18 | 9W<br>Arcade of Problems #4<br>W p. 66 | 22N<br>4.3 How Many Functions?<br>N p. 147 | 7G<br>2.1 A Grid Game<br>G p. 85 |
| 12 | 23N<br>6.4 A 7-Abacus #3<br>N p. 284 | 8P<br>1.3 Random Number Stories #3<br>P p. 25 | 10W<br>Election in the Number World<br>W p. 46 | 24N<br>5.6 Prime Factor Distance<br>N p. 222 | 8G<br>1.7 Triangle Constructions<br>G p. 47 |
| 13 | 25N<br>2.7 Guess My Rule<br>N p. 81 | 8L<br>3.2 The Table Game #2<br>L p. 90 | 11W<br>Arcade of Problems #5<br>W p. 78 | 26N<br>7.1 Cartesian Graphs #1<br>L p. 316 | ADJUSTMENT DAY |
| 14 | 27N<br>6.5 Division Algorithm #2<br>N p. 296 | 9L<br>1.5 Games with Strings #5<br>L p. 31 | 12W<br>Arcade of Problems #5<br>W p. 78 | 28N<br>5.7 Who is Gat?<br>N p. 231 | 9G<br>1.8 Breaking a Stick #1<br>G p. 53 |
| 15 | 29N<br>2.8 String Pictures with Rational Numbers<br>N p. 88 | 10L<br>3.3 The Table Game #3<br>L p. 105 | ADJUSTMENT DAY | 30N<br>5.8 Divisibility<br>N p. 241 | 10G<br>1.9 Breaking a Stick #2<br>G p. 58 |
| 16 | 31N<br>6.6 A 7-Abacus #4<br>N p. 303 | 11L<br>1.6 Games with Strings #6<br>L p. 40 | 13W<br>Election in the Number World<br>W p. 53 | 32N<br>3.3 Hand-Calculator Puzzles #3<br>N p. 116 | 11G<br>1.10 Breaking a Stick #3<br>G p. 66 |
| 17 | 33N<br>2.9 Division of Rational Numbers<br>N p. 93 | ADJUSTMENT DAY | 14W<br>Arcade of Problems #6<br>W p. 90 | 34N<br>7.2 Cartesian Graphs #2<br>N p. 326 | 12G<br>1.11 Breaking a Stick #4<br>G p. 73 |
| 18 | 35N<br>4.4 How Many Permutations?<br>N p. 157 | 12L<br>3.4 The Table Game #4<br>L p. 116 | 15W<br>Arcade of Problems #6<br>W p. 90 | ADJUSTMENT DAY | 9P<br>2.4 A Fortune Teller's Device<br>P p. 73 |

"cold", then you may be initially very reluctant to leave a lesson at the end of the time suggested, believing that several children "haven't got it yet". By keeping the spiral idea in mind, you may become more comfortable with these time schedules. Although it may appear to you that no learning is taking place in a sequence of lessons, you may be surprised that suddenly even your slower students are understanding the ideas when they encounter them later in the spiral. Whereas with a mastery approach, presented in a purely individualized mode, slow students often linger most of the school year trying to "master" low level skills (with generally poor results); the spiral approach gives these students the opportunity to benefit from the richness of the situations presented throughout the curriculum. In this manner, children are allowed to become involved in interesting and enjoyable situations which are modeled by important mathematical notions. They are not deprived of a variety of experiences just because they haven't mastered every bit and piece along the way.

4. Q: ISN'T THIS THE "NEW MATH"?

A: There is a current popular belief that all school mathematics today is either "new math" or "old math". Neither of these labels is well defined. "New math" usually denotes the programs that heavily emphasize the teaching of sets; introduce technical mathematical language for concepts such as "commutativity", "associativity", "distributivity", and so on; and that teach "concepts" before "skills". The "old math" refers to programs of the "good old days": memorization of the multiplication tables and complete emphasis on paper and pencil drill in calculations.

The CSMP curriculum is definitely not the "new math" nor the "old math" by any definition. It uses little technical language, yet achieves the understandings sought by the "new math". It uses no repetitive drills, yet elicits the ability to calculate stressed by the "old math". But these achievements are just by-products of a more pervasive goal: to help children use their curiosity, their intuition, their powers of reasoning to formulate and think about interesting problems and situations.

5. Q: IS THE STUDENT REALLY STUDYING MATHEMATICS IN THE CSMP CURRICULUM?

A: Yes. The subject matter of mathematics involves much more than calculation with numbers, and the methods of mathematics include more than the paper and pencil algorithms. CSMP believes that the number systems are very important and that it is essential to introduce students to the world of numbers. However, the intelligibility of that world is not simply a matter of computation. The CSMP curriculum is filled with experiences that directly and indirectly relate to the wide world of numbers, that support important internal relations in the world of numbers, and that encourage logical reasoning. The deep understanding of mathematics is not always verbal but is very often pictorial and CSMP tries to respect that aspect.

6. Q: WHY USE THE LANGUAGES OF STRINGS, ARROWS, AND THE PAPY MINICOMPUTER?

A: It is surprising that the language of strings was not used earlier and more extensively in the teaching of mathematics. Mathematicians have used strings for a very long time and they support notions so fundamental and natural that they go back to Aristotle. The scope of the string diagrams is broader than mathematics, the string diagram being a marvelous instrument for logical thinking. String diagrams permit the understanding of a situation at a glance.

Arrow diagrams are probably the most important teaching aid in CSMP. They are used everywhere in our daily lives: signals, advertisements, directions, and so on. Why not use them in teaching mathematics?

Arrows are a pictorial way to represent relations. Relations have always been important in mathematics, but in the past have seemed to be something rather abstract, elusive, and mysterious. With arrows they become vivid, concrete, and easy to understand.

The Minicomputer allows very young children to become familiar with numbers and operations on them (especially large numbers and the more difficult calculations) before they are able to work with them on paper. Later, and more importantly, the Minicomputer is used as a vehicle for posing situations involving mental arithmetic and for examining the workings of the arithmetic operations. It provides the experience necessary so that rather than memorizing an algorithm and applying it to situations, the situations give rise to the algorithm.

7. Q: IF THE STUDENTS USE THE MINICOMPUTER, WILL THEY BE ABLE TO CALCULATE WITHOUT IT?

A: This question suggests that the only purpose of the Minicomputer in the CSMP curriculum is to perform calculations. To be sure, the Minicomputer does enable very young children to deal with "large" numbers and to perform relatively difficult calculations long before they can write the paper and pencil algorithms for the calculations. But very soon these experiences with the Minicomputer suggest how the paper and pencil algorithms come about, and from that moment the Minicomputer is no longer regarded as a calculating tool. Instead, for the remainder of the curriculum, the Minicomputer assumes its primary roles: as a source of very interesting problems and games, as an effective vehicle for stimulating mental arithmetic, as a teaching device for introducing more advanced topics such as negative numbers and decimal numbers, and, in general, as an alternate representation of numbers.

8. Q: WHAT IS THE CSMP POSITION ON HAND-CALCULATORS AND WHAT ROLE DO THEY PLAY IN THE PROGRAM?

A: Hand-calculators should be a part of any mathematics curriculum. As more and more children have hand-calculators, these tools will become incorporated in the elementary and secondary curricula in a natural way with uses very similar to those of the Minicomputer. In some cases, they may be used as a tool for mechanical computation where the difficulty of the calculation might turn an otherwise interesting problem into an exercise in drudgery. In other cases, they may be used as the context for posing situations leading to the discovery of important mathematical properties or providing opportunities for interesting exercises in mental arithmetic.

Hand-calculators are of particular benefit to "slow" learners. Many slow learners have mechanical difficulties rather than problems in thinking about a situation. The hand-calculator alleviates this problem.

Some lessons in the CSMP curriculum require only two calculators, while others suggest that each student have one. Sometimes lessons of this latter sort can be modified so that students may share hand-calculators or may simply think about how a hand-calculator works. For some lessons, however, every student should be provided with a hand-calculator. In either situation, students with access to hand-calculators should always be encouraged to bring them to class.

9. Q: WHY IS THERE A LACK OF DRILL AND LITTLE EMPHASIS ON "SKILLS"?

A: CSMP believes that training for rote skill should never precede understanding. The avowed purpose of drill in arithmetic is to acquire the "skills", that is, the skills of calculation. Certain numerical skills are essential, but, unfortunately for school children, the usual drill techniques for acquiring them are dull, stultifying, and sometimes even counterproductive. The CSMP curriculum appears to place little emphasis on "skills" because it does not resort to drill. But as a matter of fact, the essential skills of both mental and written calculation are learned almost inadvertently in response to the many situations where the ability to calculate is needed in order to respond to the situation: to win the number game, to identify the secret number of the detective story, or to build a number road. Calculation is always considered as a tool for doing interesting things, rather than as a chore done for its own sake. See also the comments on Evaluation, Section 2 (d) of the General Introduction.

10. Q: HOW DO STUDENTS LEARN TO USE MATHEMATICS IN PRACTICAL SITUATIONS?

A: For students of elementary school age, a situation that intrigues them and catches their imagination or their curiosity will certainly motivate them to learn whatever is necessary in order to deal with the situation. But situations that adults consider "practical" very often are outside the realm of interest of school children and do not provide motivation for learning. CSMP has found that students will indeed learn to use mathematics in situations they consider worthy of their attention. As they approach adulthood they soon will consider "practical" situations to be most worthy of their attention, but in the meanwhile, they are more interested in games, fantasy, puzzles, stories, and so on.

11. Q: WHY ARE WE TEACHING PROBABILITY?

A: Next to arithmetic, probability has the most direct application to life in such areas as economics, meteorology, data processing, insurance, most of the natural sciences, and most of the social sciences. Probability differs from many areas of mathematics in that it is applied directly. One does not need to combine knowledge of probability with any deep theoretical knowledge in order to apply it to "real world" problems. One encounters statistics nearly every day of one's life, and the better developed one's intuitive sense of probabilistic situations is, the less likely one is to be "duped" by statistics.

12. Q: HOW ARE TELLING TIME, MONEY, AND THE METRIC SYSTEM TREATED?

A: Telling time is not included in the CSMP curriculum. It is our belief that telling time is more naturally a part of a language program than of a mathematics program, although it is frequently included in the mathematics curriculum. Since there are many excellent materials available to help you teach telling time, we urge you to teach this topic using those methods and materials you particularly like or that have proved successful for you in the past.

The topic of money is first introduced in the CSMP curriculum in lessons involving combinations of coins. It is treated again in the Upper Primary Grades and the Intermediate Grades as a support for the teaching of decimal numbers. Although money is not treated as a separate topic, these situations provide considerable experience with our monetary system. For students who need more exposure to this topic, teachers should feel free to develop supplementary materials in order to provide extra practice.

Metric units are used whenever there is an occasion to measure in standard units. Activities involving the metric system are begun in the First Grade Program and continue throughout the Geometry and Measurement strand of the Upper Primary Grades Program and the Intermediate Grades Program. We recommend that you do not ask the students to convert from the English to the metric system or vice versa (inches to centimeters, yards to meters, Celsius to Fahrenheit, grams to ounces, and so on). Since the United States has not yet "gone metric", your students will still need to be familiar with the English system. We suggest that you use whatever materials and methods you have found successful in the past for teaching this topic, but we repeat our recommendation that you not ask the students to convert from one system to the other.

13. Q: WHAT IS THE ROLE OF STORYBOOKS AND STORY-WORKBOOKS IN THE PROGRAM?

A: The storybooks in the collection STORIES BY FREDERIQUE include twenty-four stories for children, presented in colorful newsprint booklets. They are intended to appeal to children in the 5-14 age bracket. The stories are completely independent of each other and need little or no external explanation. They may be used for individual or collective reading, at home or at school. The story-workbooks, as the name implies, are a cross between storybook and workbook. They present more problems for the student to solve then do the storybooks, but within a story setting. The story-workbooks comprise the collection called THE CSMP LIBRARY. Each of the stories is portrayed by the mathematical languages of dots, arrows, and strings which reinforce the ideas of the stories. A young reader can become personally involved in the stories; there are questions to answer, pictures to relate to the stories, and new adventures to invent for the characters in the stories. The reader discovers that emotions and aesthetic senses can be involved in the world of mathematics just as in the world of people.

Among the goals of the various storybooks and story-workbooks are the following: to familiarize the students with nonverbal mathematical languages and to leave them with the impression that these languages are as natural and useful as the usual language; to introduce some mathematical games; to present detective stories which introduce some interesting mathematical problems; to provide reinforcement for all students and especially for the slower learners.

14. Q: WHAT PROVISION IS MADE FOR INDIVIDUALIZATION IN THE CSMP CURRICULUM?

A: In many of the lessons there are worksheets to be done individually by the students. Sometimes the worksheets are done at the beginning of a lesson, followed by a full group discussion of them. But ordinarily a topic is first developed in a full group setting, followed by individual student work on related worksheets. If "individualization" means a precedure for allowing students to work alone at their individual rates and levels, then the worksheets (and also the workbooks, discussed below) do provide for (partial) individualization. A series of worksheets for a given lesson offers a whole spectrum of levels from very simple to quite challenging. Students all start with the first worksheet (one-star) and proceed at their own pace and levels as far as they are capable. The spiral approach accommodates this kind of individual work in the same manner as the full group work; that is, no worksheet is a prerequisite for the content of any succeeding lesson.

Twelve lessons during the semester are devoted to the six IG-V workbooks. Each workbook is also designed to move through a spectrum of levels of difficulty. We believe that the worksheets and workbooks realize the best sense of "individualization".

15. Q: WHAT PROVISIONS ARE MADE FOR THE VARIETY OF ABILITIES IN A USUAL CLASS?

A: Every lesson includes something for all students; there are opportunities for the slowest students to participate in a meaningful way, and there are challenges for the brightest students. This means that you should not exclude any students from a full group lesson. At the same time, you should not expect every student to understand everything during the course of the lesson. Furthermore, with the spiral development many situations are open-ended and allow students to experience them at levels as high as they are capable.

Evaluation has shown that students traditionally labeled "slow" are doing better in CSMP classes than in non-CSMP classes. We believe this is due to the spiral development, to full group work on mathematical activities, and to the presence in the CSMP curriculum of certain nonnumerical activities in which many supposedly "slow" students can excel.

The slow learners should participate in math with the other students in the class, but they might also benefit from additional small group work. The additional time should not be simply to rehash or drill on the situations presented in class, but rather to provide additional experiences for these students. See the book "Math Play Therapy" Volumes I and II for suggestions of small group work with slow learners.

Additionally, CSMP provides for a variety of ability levels in the student materials. Worksheets and workbooks offer problems in a range from very easy to quite challenging. Indeed, some of the "higher starred" worksheets and pages toward the end of the workbooks may be accessible only to your very best students.

In an extreme case, you might seriously consider the possibility of arranging for exceptionally bright students to take their math with a higher grade level CSMP class.

16. Q: IS IT NECESSARY TO FOLLOW THE SUGGESTED SCHEDULE EXACTLY?

A: This program is not meant to be a static sequence of exercises to be followed rigidly. The lessons in this guide are suggestions and the time allotments are merely approximations based on our own experience teaching the materials. If this will be the first time you are teaching the program and you would like specific suggestions for organizing the lessons, we offer some guidelines and a suggested schedule of lessons. You may wish to follow it at the beginning of the school year and then deviate from it as the needs of your students dictate.

It is probably best to follow a one-day, one-strand pattern when you organize your own schedule, rather than to spend two or three full weeks "covering" one particular topic. Interchange the strands and the days of the week as you please. In a usual five day week, two days should be devoted to the World of Numbers strand, one day to the Workbook strand, and the other two days should be shared by the P, G, and L strands.

Exercise your own professional judgment in deciding when to stop a particular lesson. It is better to stop a lesson when student interest is high then to teach so methodically that everyone in the class understands everything, but is completely bored. In order to be successful with this curriculum, adjust the schedule to the ability, maturity, and interests of your class.

17. Q: HOW DO I HANDLE STUDENTS WHO HAVE AN EXTENDED ABSENCE OR STUDENTS (NEW TO CSMP) WHO TRANSFER INTO MY CLASS DURING THE YEAR?

A: The spiral approach allows a student to miss one or two weeks of school without much "catch up" work needed. When the student returns, either you or classmates can help with any material that is deemed necessary.

If several months of classes have been missed, you might need to treat the student as you would a transfer student new to CSMP. Give the student a little extra attention during a lesson especially when the questions are review for most of the class.

In most cases, the differences between new and veteran students tend to fade rather quickly. One exception might be in lessons involving the Minicomputer. Veteran students who are unusually adept with the Minicomputer very often are good tutors for the transfer students in the use of this abacus. In addition, the storybook TWO BY TWO is suggested as an aid for new students who are unfamiliar with the Minicomputer. Some of the storybooks and story-workbooks written for younger children can be used to introduce new students to the languages of strings and arrows.

It is probably best not to send worksheets and/or workbooks home to a child. Without participating in the collective lessons that introduce the worksheets and workbooks, it would be very difficult for the student to proceed.

18. Q: HOW DO I HANDLE WHOLE CLASSES WHICH ARE NEW TO CSMP?

A: CSMP does not assume that kindergarten and first grade students have any prior experience with CSMP. The Kindergarten and First Grade Programs will need no adaptation for classes which are new to CSMP.

Second grade classes which are new to CSMP will need to be introduced to the CSMP languages. For this purpose, a suggested adaptation of certain second grade lessons is included in the CSMP Mathematics for the Upper Primary Grades (UPG) Part I teacher's guide.

Third grade classes which are new to CSMP will need to become fluent in the CSMP languages before beginning many of the lessons in UPG Part III. A special schedule has been prepared for these classes which consists of prerequisite lessons and workbooks from UPG Part I and II followed by a modified UPG Part III and IV program.

Fourth grade classes new to CSMP spend one semester with CSMP Mathematics for the Intermeidate Grades (IG), Entry Module I, before beginning the regular fourth grade program. Similarly, fifth grade classes new to CSMP spend approximately one semester with an Entry Module II before beginning a modified fifth grade program.

19. Q: WHAT SHOULD I DO WITH MY CLASS IF THEY DID NOT FINISH THE PREVIOUS YEAR'S SCHEDULE?

A: No special measures are required. Each part of the CSMP curriculum begins with introductory lessons. In addition, the spiral approach permits individual participation at the student's own level of understanding, even if that level is not quite as high as it might have been if the previous year's schedule had been completed.

20. Q: HOW CAN I ASSESS STUDENT PROGRESS AND ASSIGN GRADES?

A: CSMP does not believe that it is either desirable or reasonable for teachers to attempt to evaluate student mastery of skills from day to day or even from week to week. On the other hand, teachers, parents, and even the children themselves benefit from some monitoring procedure that records involvement in activities and assesses progress through the program.

Most teachers have their own monitoring devices with which they are comfortable. It is recommended that such devices be continued.

21. Q: HOW DO CSMP STUDENTS DO ON STANDARDIZED TESTS?

A: While CSMP disagrees with the philosophy underlying standardized testing, test results are available to us from several sources. The results indicate that CSMP classes generally do as well or better on tests than do non-CSMP classes. It should be pointed out that standardized testing and traditional programs are correlated and that the CSMP student has been exposed to a great deal of mathematics that is not taught in more traditional programs or considered in the construction of standardized tests. There are also many unmeasurable or unmeasured areas such as the enjoyment of mathematics, strategic thinking, organizational ability, and depth of understanding of the methods and uses of mathematics where we feel CSMP students have a definite advantage. See also the comments on Evaluation in Section 2 (d) of the General Introduction.

22. Q: WHAT PROBLEMS WOULD CSMP STUDENTS HAVE IF THEY RETURN TO A MORE TRADITIONAL PROGRAM IN THE NEXT SCHOOL YEAR?

A: There should not be problems for CSMP students who return to a traditional program. Test results indicate that CSMP students are at or above the level of non-CSMP students in material covered in traditional courses. In addition, CSMP students have encountered many situations that are not introduced in traditional programs nor tested for on standardized tests. (See Question 21) In particular, their reasoning ability should be much sharper than their non-CSMP counterparts.

There are, of course, differences in timing as to when different programs introduce the standard algorithms and often CSMP's introduction will occur later than in some of the other programs. The reason for this is our belief that it is essential that the creative freedom of students not be impaired by a premature emphasis on the standard algorithms for adding, subtracting, multiplying, and dividing. We believe that many of the present computational difficulties experienced by elementary and junior high school students were caused by their spending an inordinate amount of time trying to master the algorithms too early in their mathematical education. This shuts the students off from other experiences necessary for their mathematical growth. Students need plenty of time to explore the world of numbers. Mastering an algorithm should be the last step of this exploration, a sort of synthesis of previous experience. Furthermore, as electronic hand-calculators become available to nearly everyone, algorithmic mastery as a prerequisite to other numerical experiences should become less important.

CSMP students can usually solve a problem before having learned the standard algorithm. If they enter a program that expects an algorithm they have not learned, they are still more than adequately prepared for the algorithm and should have no trouble quickly learning to use it.

CSMP introduces the usual addition algorithm in the second grade, one of the two standard subtraction algorithms in the third grade, the multiplication algorithm in the fourth grade, and the division algorithm in the fifth grade.

23. Q: WHAT ARE SOME WAYS TO ORGANIZE MATERIALS AND MANAGE MY CLASS?

A: It is very important to organize your CSMP materials before school begins. Decide where you will store the materials once they are organized. Filing drawers, shelves, boxes, or closets are all satisfactory as long as you can easily get to the materials. Take them all out of their shipping boxes and spread them on several tables and/or desk tops in your classroom so they can be organized into the order you will be using them.

Workbooks: The teacher's guide is needed to find the order in which the workbooks are used. Locate the Workbook strand column in the Master Suggested Schedule of Lessons and collect all the workbooks in the order in which they are listed. After organizing the workbooks in order, place them in an upright position in your file drawer, box, shelf, etc., so that the first workbook to be used is at the front of the stack.

Worksheets: There is a separate worksheet booklet for each student and you should decide whether you will store these booklets and distribute them when a particular lesson calls for a worksheet, or whether you will simply let the students keep their own worksheet booklets in their desks.

Each worksheet can be removed from the booklet or left in depending on how you wish to manage worksheets. However, once a worksheet is removed from the booklet it cannot be replaced, so make sure students are careful about tearing off the correct sheet.

Manipulatives: Storage of each manipulative in a separate container, properly marked, is one way to make everything accessible. Another way is to provide a sturdy box (for example, a shoe box) for each student and to put individual manipulative materials in the box, properly labeled with the student's name.

Additional Materials: Be sure that you have plenty of high quality colored chalk, large sheets of unlined white paper, and that your students have colored pencils, pens, or crayons. Plan well ahead to be sure you have any needed materials that are not supplied by CSMP.

Keep all teacher materials where they are easily accessible. You may find the guides easier to use if they are put in looseleaf binders; the books are perforated so that pages can be removed easily. The demonstration Minicomputer boards and a number line should always be on display in the room.

24. Q: SOME OF THE CSMP MATERIAL IS NEW TO ME. HOW DO I LEARN ENOUGH TO TEACH A CLASS?

A: Every CSMP teacher should have been trained by his or her coordinator. Kindergarten and first grade teachers receive a minimum of eighteen hours training, teachers of the Upper Primary Grades Program receive up to thirty-two hours training, and teachers of the Intermediate Grades Program receive up to fifty-six hours training. During the training sessions the CSMP content is taught to new teachers and suggestions are given for organization of their classes.

If the suggested training schedule has been followed, many lessons taken from the CSMP curriculum, or very similar lessons, have been presented. The training schedule also makes use of the spiral approach so that training sessions aim not only to teach CSMP mathematics but to familiarize teachers with the spirit and pedagogy of the CSMP curriculum.

25. Q: WHAT SHOULD BE DONE WHEN THERE IS A SUBSTITUTE TEACHER?

A: Some school systems have included people who frequently substitute in their CSMP training program. When this is not the case, it might be very difficult for the untrained substitute to teach from CSMP materials. One solution is to arrange for another CSMP teacher to teach math to your class while the substitute teacher teaches another subject to that teacher's class. If that can not be arranged and you will only miss one or two days, you may find certain CSMP material, uncompleted workbooks for example, that the substitute can manage. Otherwise, use those days to cover material not included in the CSMP curriculum, such as telling time. If your absence will be an extended one, arrangements should be made with your coordinator to train a substitute, rearrange schedules to put a CSMP teacher in your class, or otherwise provide for your students.

26. Q: THE PROGRAM MAKES USE OF THE STUDENT'S CREATIVE FREEDOM. HOW MUCH ROOM IS THERE FOR THE TEACHER'S INGENUITY?

A: CSMP not only allows for your ingenuity but depends on it. While the lessons may appear to be very rigid by presenting dialogue for teacher and student, it should be remembered that these are suggested questions, answers, and comments based on our experiences with development classes. Even if you were to follow your part of the "script", your students are likely to offer responses which differ from those we have listed. It is the students' responses that will determine the content and direction of your next question or comment.

In a traditional program your creativity is called upon the first time you develop a particular lesson, and to a lesser extent, as you revise lessons in succeeding years. There is little room for spur-of-the-moment creativity once a lesson begins. The CSMP curriculum, by creating a free and open situation emphasizing the student's creative freedom, continually presents you with unanticipated student responses and calls on your ingenuity at all times.

27. Q: HOW CAN I EXPLAIN CSMP TO PARENTS?

A: The best way would be for your coordinator to organize a workshop for interested parents. The parents can then see the languages of the program in use and have some of the same experiences their children have. Workshops can be of almost any length but anything short of two hours does not allow for much familiarity with the material. In such a workshop, we suggest that the language of the Minicomputer be explained with particular emphasis on its use in presenting situations where mental arithmetic is called upon and in developing algorithms (for example, physically carrying a checker from the ones' to the tens' board in the "8 + 2 = 10" play, paralleling "carrying" in the paper and pencil algorithm). A lesson using the storybook WHERE'S MY NOSE? provides a good example of the power of the language of strings. A nonnumerical example, such as the brother-sister graph, and a numerical example, such as building a road from 5 to 55 using +3 and 5x arrows, will provide an introduction to the language of arrows. The lessons that are the favorites of the children would be excellent for a presentation to parents as well.

Also you might extend an invitation to parents to visit your class during the math period. Many times such an invitation leads eventually to a closer parent-teacher relationship, with the parent offering voluntary services as an aide.

You will find a letter to parents at the end of this section which may be reproduced and sent home with the students.

## TO THE PARENTS OF CSMP STUDENTS

Your child is participating in the Comprehensive School Mathematics Program's (CSMP) elementary school curriculum, developed under a contract with the National Institute of Education. While the program cannot be fully explained in a brief note, we hope that this letter will give you some feeling for the substance of your child's mathematics education.

The content goals of the CSMP K-6 curriculum include a development of the standard number systems and their operations (including an early introduction to the integers, rational numbers, and decimal numbers) interwoven with a development of geometry and measurement. On an equal footing with the above, the CSMP curriculum includes a strong component of probability, statistics, and combinatorics which extends throughout grades 2-6. Powerful pedagogical devices are used, such as several nonverbal languages. These make the unifying ideas of the program accessible to very young children.

These languages include:

1) The Language of Strings This "language" mimics the basic methods of collecting and classifying data related to the problems at hand. That is, it deals with the fundamentally useful and important mathematical notion of set.

2) The Language of Arrows This graphical "language" models the process of comparing and analyzing sets and operations on them. That is, it deals with the mathematical notions of relations and functions including, but not restricted to, ordering relations and the numerical processes of adding to, subtracting from, multiplying by, and sharing equally among.

3) The Language of the Papy Minicomputer The Papy Minicomputer, an extremely effective paper abacus invented by Georges Papy, models the positional structure of our system of numeration and lends itself to a multitude of numerical algorithms, including the standard algorithms for the basic numerical operations.

Among other things, these languages allow for the development of the essential operations on numbers in a natural way based on previous experiences with the languages. They also provide a context in which situations arise that call for the repeated use of arithmetic operations without resorting to drill exercises, which most children find boring.

Another tool, not unique to but used by CSMP, is the hand-calculator. The hand-calculator allows the child to study problems that call for calculation beyond the child's ability or in which the calculation is so extensive as to cause the child to lose interest. The calculator is also used in a way similar to the use of the languages mentioned above. Properly used, it can help lead to the development of important mathematical ideas and present situations for the interesting use of mental or "paper and pencil" arithmetic.

The content is learned in an atmosphere of constant applications utilizing a "pedagogy of situations".* These vary from simply story situations in the early primary grades to challenging applications and nontrival simulations of real world problems in the upper grades.

One of the tenets of CSMP's development philosophy is that mathematics is a unified whole and should be learned as such. As a result, along with CSMP's concern with the variation in ability levels in the typical heterogeneous class, the curriculum is structured in a spiral format throughout the K-6 program. The content is completely sequenced in spiral form so that each student is brought into contact with each area of content continuously throughout the program. This consciously avoids atomizing the content and mastering each bit before continuing to the next. Rather, each student spirals through repeated exposures to the content, building interlocking experiences of increasing sophistication.

Another tenet of CSMP's development philosophy, based on years of experimentation in the classroom, is that no single method of classroom management suffices to meet the needs of every student. Hence, the program is constructed to allow numerous opportunities for whole class participation, small group participation, and individual independent experiences. Individualization throughout the K-6 program is achieved through the independent work on workbooks and worksheets of varying levels of difficulty. Due to the spiral nature of the curriculum, each student can engage in a workbook without losing touch with the progress of the class while at the same time working to the limits of his or her ability.

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* Inspired by the work of the mathematician and psychologist, Caleb Gattegno.

A feature unique to the CSMP program is a collection of storybooks written by Frédérique Papy (STORIES BY FREDERIQUE) and a collection of story-workbooks (THE CSMP LIBRARY). These probe the affective domain of learning and bring children into contact with areas of mathematics until now unexplored by them.

CSMP does not have its own system of student assessment and progress reporting. While standardized testing indicates that CSMP students develop the "basic skills" as early or earlier than do non-CSMP students, we hope you will not be anxious about your child's development of these "skills". Each child is an individual and should not be judged against norms but on what he or she can and does accomplish. Many extremely important areas -- development of self-confidence, ability to think and reason, etc. -- are very difficult to measure and grade. If you have any questions regarding the progress of your child, his or her teacher is in the best position to answer those questions.

THE CSMP STAFF

# THE WORLD OF NUMBERS

Comprehensive School Mathematics Program

# Mathematics
# for the
# Intermediate Grades
# Part V

## The World of Numbers

CEMREL, Inc.

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# THE WORLD OF NUMBERS

## INTRODUCTION

By this time, your students have had a rich variety of experiences with various kinds of numbers and operations and relations on them. These include encountering both positive and negative integers, decimal numbers, some rational numbers, numerical functions (e.g., $2x$; $+3$; $\div 10$; $-5$; $\frac{3}{4}x$), order relations (e.g., $<$ and $>$), and the notions of multiples and positive divisors of a given number. They have been introduced to paper and pencil algorithms for addition, subtraction, and multiplication of whole numbers and decimal numbers, and they have had extensive experiences using systematic methods for division and for multiplication and addition of rational numbers (fractions) in preparation for algorithms.

In the present strand, these previous numerical experiences will be revisited, extended, and deepened. New situations will be presented, including the introduction of several fascinating extensions and modifications of numerical games: Guess My Rule (Section 2.7, Section 5.1, and Section 5.5); Hand-Calculator Puzzles (Chapter 3); Binary Codes (Sections 4.1, 4.2, and 4.3); and Detective Stories (Chapter 1 and Section 5.7); and new numerical games: A 7-Abacus (Sections 6.1, 6.2, 6.4, and 6.6) and Cartesian Graphs (Chapter 7). Old and new concepts will be explored in the context of these situations.

The first paragraph above, describing some of your students' previous numerical experiences, simply catalogues some of the realms of the world of numbers your students have visited. It does not indicate the travel routes nor the vehicles used to travel them.

The CSMP curriculum stresses the unity and continuity of the growth of mathematical ideas and concepts. This is reflected in numerous ways in this strand, as in the others. For example, CSMP employs the spiral approach, not pausing for mastery of each lesson but rather allowing students to encounter the elements of each content strand in different situations, interspersed with other strands throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, CSMP presents the content in a situational framework. That is, a "pedagogy of situations" is employed which engages students in rich problem solving activities. These offer the opportunity both to develop necessary numerical skills and to gain deeper understanding of mathematical ideas and concepts in the world of numbers. At the same time, the situations presented encourage students to develop patterns of logical thinking and strategies for attacking problems.

Perhaps the most important embodiments of the CSMP approach are the three nonverbal languages used throughout its elementary program: the Papy Minicomputer, the language of arrows, and the language of strings. These are the vehicles which make it possible to develop the situations in which the content is presented to students. Further, as students grow in the CSMP curriculum, these languages become basic investigatory tools available to explore the content.

It is hard to overstate the value of developing tools which aid in attacking problems as well as in representing situations; tools which are not confined to one area of mathematical content or to one level of the development of content. In the usual elementary curriculum, students must continually learn new languages to express each new piece of mathematics that is studied. Armed with the universally applicable languages of the CSMP curriculum, the students grow more and more familiar with the syntax of these languages and are freed to explore new content directly in terms of these same languages rather than being forced to think of each new mathematical idea as tied to a certain new language. This is definitely not to say that the CSMP students do not learn the usual descriptive language of mathematics. However, in the CSMP approach the usual descriptive language is not the vehicle for learning, but only a means for describing the ideas.

For specific examples of the situations and the ways CSMP uses nonverbal languages in this strand, we refer you to the short introductory remarks preceding each chapter and, in particular, to the lessons themselves.

## A WORD ABOUT STANDARD ALGORITHMS

We mentioned above that CSMP seeks to develop the necessary skills in the numerical domain as well as deeper understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that along with the growth of understanding in the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and the operations and relations on them. But facility must not be confused with understanding. They are partners in the growth of mathematical maturity, and it is extremely important that neither be sacrificed for the other. Rather, a balanced growth in each must be maintained.

In the usual elementary curriculum, paper and pencil algorithms for the four basic operations provide a prime opportunity for the abuse of the position stated above. It is undeniably important that a student eventually learns mechanical algorithms for the basic operations. However, if these algorithms are presented prematurely they stunt the student's ability to develop alternative algorithms, to do mental arithmetic, to estimate, and so on.

For example, rather than employing the usual division algorithm to determine the standard name for 1,791 ÷ 9, it may be easier and more efficient to note that 200 x 9 = 1,800, so that 199 x 9 = 1,791 and 1,791 ÷ 9 = 199. Indeed, built into this way of approaching the problem is a relatively good estimate (1,800 ÷ 9) of the number involved.

If we insist on a mechanistic response to problems such as the one above (i.e., insist on the use of paper and pencil algorithms), we may not only encourage inefficiency in many situations but also may inhibit the development of flexibility necessary for problem solving. On the other hand, if we present to students a rich array of situations in which they interact with numbers and their operations and let the standard algorithms grow naturally as just one facet of the students' experience, we obtain two important results. The students will gain the necessary facility with standard algorithmic procedures and at the same time retain the openness required to respond creatively to new situations in the world of numbers.

## NOTE ON THE USE OF HAND-CALCULATORS*

The hand-calculator provides another powerful tool for the investigation of problems and situations in the world of numbers. This tool is intimately related to the Papy Minicomputer since its internal structure is an electronic simulation of the Minicomputer. Although this analogy need not be acknowledged by you or your students, it does serve to indicate that the hand-calculator is indeed a natural tool in the CSMP curriculum.

Hand-calculators are recommended materials for Part V of the Intermediate Grades Program. They are not included in your complete classroom set; however, they may be purchased from CSMP as separate items. There are extensive opportunities for the use of hand-calculators in the lessons of Part V, especially in Chapters 1 and 3 of this strand. In fact, some lessons require each student to have access to a hand-calculator and would be practically impossible to teach otherwise. The workbooks present, by way of review, situations involving the hand-calculator and can be understood only from previous hands-on experience with calculators. In addition, the supplementary material for the slow learner, "Math Play Therapy", relies heavily on the use of hand-calculators.

We are firmly convinced through our own experience with children that the hand-calculator is a potent pedagogical tool when used properly. The time is not far off when its presence in the classroom will be as commonplace as the pencil. For the present, CSMP will continue to investigate the use of hand-calculators in the classroom, particularly with slow learners, and will develop situations designed for their use.

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* See General Introduction, Section 2(f) for a more detailed discussion on hand-calculators.

# 1

# DETECTIVE STORIES AND COMPOSITION

In this first chapter of the World of Numbers strand, many of the ideas from previous parts of the Intermediate Grades curriculum are reviewed in the context of detective stories. Composition of numerical functions is reviewed in various arrow diagram settings. Attention is focused on the distinction between composites which involve only functions of the form "$\boxed{\phantom{0}}$x" or "÷$\boxed{\phantom{0}}$", or which involve only functions of the form "+$\boxed{\phantom{0}}$" or "-$\boxed{\phantom{0}}$", and composites which involve some combination of multiplication, division, addition, and subtraction functions. Observe that in the former case a commutative property is in effect, whereas in the latter it is not. Begin an investigation of a distributive property in situations involving composites of multiplication (or division) and addition (or subtraction) functions. For example,

$$3 \times (\text{Flap} + 40) = (3 \times \text{Flap}) + 120$$

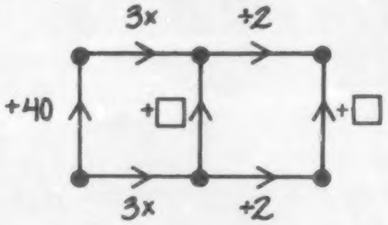
so, +40 followed by 3x is the same as 3x followed by +120.

CAPSULE LESSON SUMMARIES

Section 1.1 Decode configurations on the Minicomputer. From a list of numbers, select those that can be put on the Minicomputer with exactly one of the checkers: ②, ... , ⑨. Solve a detective story in which clues involve patterns generated by combined use of the hand-calculator relations $\boxed{+}\ \boxed{5}\ \boxed{=}$ ... and $\boxed{+}\ \boxed{3}\ \boxed{=}$ ... , numbers which can be put on the Minicomputer with weighted checkers, and a pattern generated by using the hand-calculator relation $\boxed{-}\ \boxed{4}\ \boxed{=}$ ....
(Worksheets N01, N02)

Section 1.2 Use a string picture to review the relation $=_3$ and observe that two numbers are equal modulo 3 if and only if their difference is a multiple of 3. Solve a detective story in which clues involve writing numbers by inserting parentheses into the expressions "6 + 4 x 5 - 3" and "8 x 4 + 6 + 2", the relation "is a positive divisor of", and the relation $=_3$.
(Worksheets N03, N04, N05, N06)

Section 1.3 Label the arrows in an arrow picture involving arrows for multiplication functions. Using composition of addition and multiplication functions and of addition and division functions, label arrows in a picture such as :



With the hand-calculator as support, label arrows which complete a cycle of addition, multiplication, and division arrows.
(Worksheets N07, N08, N09, N10)

## 1.1 Who is Tob?

CAPSULE LESSON SUMMARY

Decode configurations on the Minicomputer. From a list of numbers, select those that can be put on the Minicomputer with exactly one of the checkers : ② , ... , ⑨ . Solve a detective story in which clues involve patterns generated by combined use of the hand-calculator relations $\boxed{+}\ \boxed{5}\ \boxed{=}$ ... and $\boxed{+}\ \boxed{3}\ \boxed{=}$ ... , numbers which can be put on the Minicomputer with weighted checkers, and a pattern generated by using the hand-calculator relation $\boxed{-}\ \boxed{4}\ \boxed{=}$ . . . .

MATERIALS

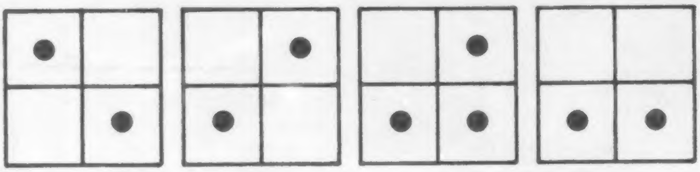
Teacher : Minicomputer set[†]; weighted checkers; colored chalk
Student : Paper; pencil; hand-calculator; Worksheets N01 and N02

### DESCRIPTION OF LESSON

Exercise 1

If you prefer, instruct students to perform the decoding of numbers on the Minicomputer with hand-calculators rather than with paper and pencil.

Display four Minicomputer boards and slowly put on this configuration.



T : Write this number on your paper.

S : 9,673.

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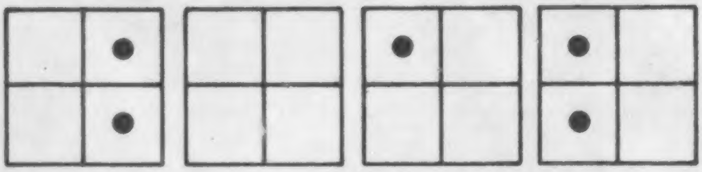
[†] A teacher's Minicomputer set consists of four demonstration Minicomputer boards and a sufficient number of demonstration Minicomputer checkers.

# N 1

Put this configuration on the Minicomputer and ask the students to decode it on their papers.



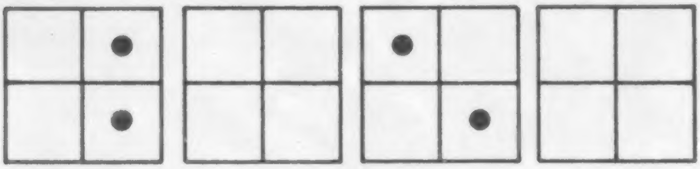
= 5,090

For students who need to review decoding, ask :

T : Can you show the same number using one checker less?

Invite a student to make a play.

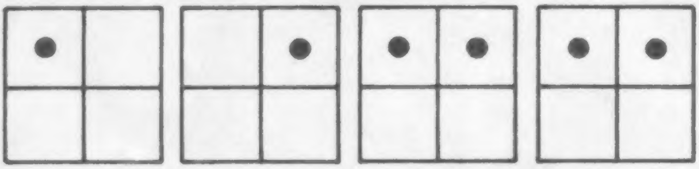
S : 8 + 2 = 10.



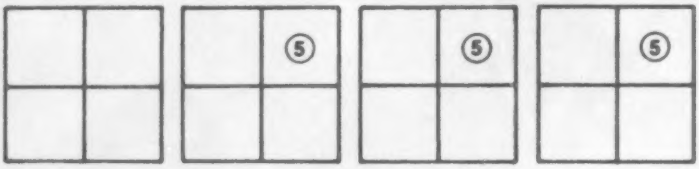
T : Which number is on the Minicomputer?

S : 5, 090.

Continue the decoding with the following configurations. For students who have difficulty decoding, use a questioning method similar to the previous one.



= 8,532



= 2,220

## Exercise 2

Display two Minicomputer boards and these checkers :

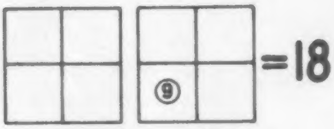
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

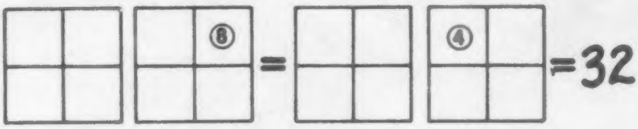
Write this list of numbers on the chalkboard.

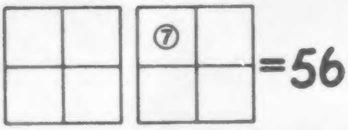
18 32 54 56 240 400 420

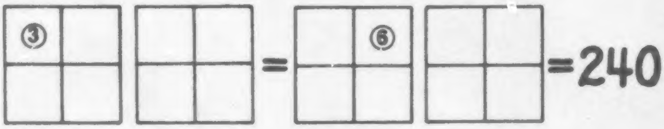
T : Which of these numbers can you put on the Minicomputer using exactly one of these checkers — ② through ⑨ ?

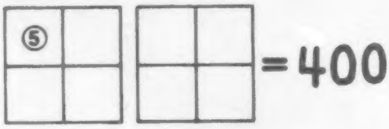
Ask a student to put a number on the Minicomputer, and then direct the rest of the class to check that the number on the Minicomputer is one of the numbers in the list. Continue until the class finds that these five numbers — 18, 32, 56, 240, and 400 — can be put on the Minicomputer using exactly one of the displayed checkers. The next illustration shows one or more ways for each number.









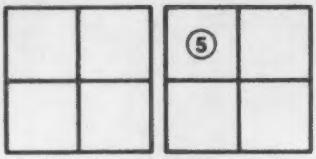


The class should conclude that 54 and 420 cannot be put on the Minicomputer with exactly one of the displayed checkers.

# N 1

T: On your paper show how 54 can be put on the Minicomputer using a ⑤-checker and a ⑦-checker.

Let the students work independently for a few minutes; then ask a student to place one checker on the Minicomputer. Suppose it is a ⑤-checker.



T: Which number is on the Minicomputer?

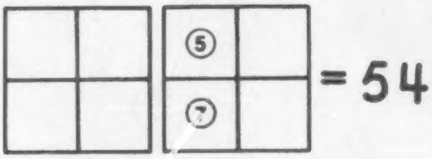
S: 40, because 5 x 8 = 40.

T: How much larger is 54 than 40?

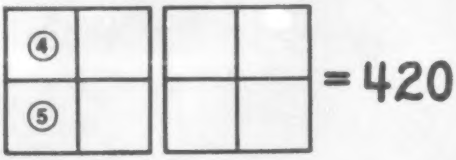
S: 14 larger.

T: Can someone put 14 on the Minicomputer using a ⑦-checker?

Invite a student to place the ⑦-checker.



Continue by instructing the students to show 420 on the Minicomputer using a ④-checker and a ⑤-checker.

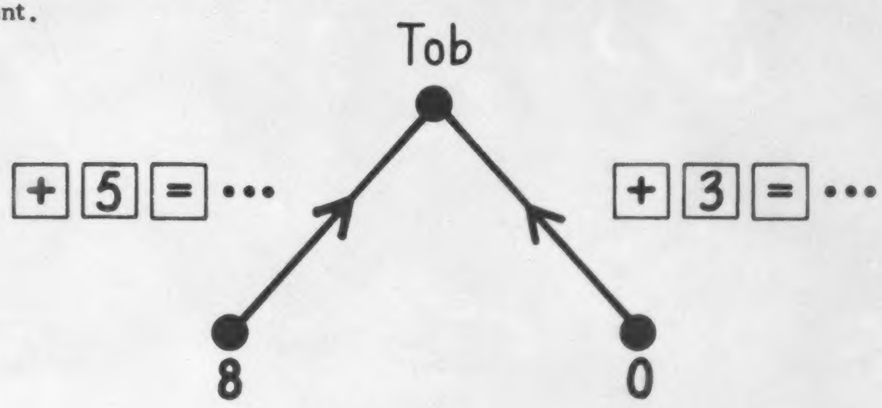


## Exercise 3

Tell the class that they are going to solve a detective story about Tob.

**First clue**

Draw this arrow picture on the board and provide a hand-calculator for each student.



T : Tob is in this arrow picture. Which numbers could Tob be?

Give the class a few minutes to explore this situation.

S : Tob could be 18.

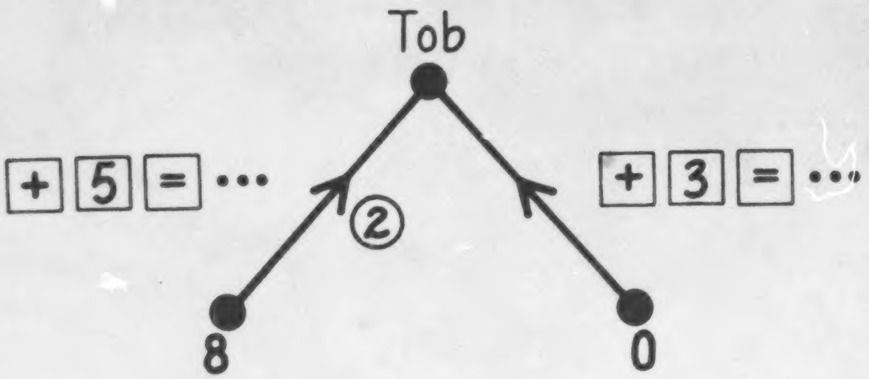
As the students find numbers that Tob could be, list them on the board. Occasionally, ask the class to verify that Tob could be a particular number. For example,

T : Tob could be 18. If we start from 8 and press $\boxed{+}$ $\boxed{5}$, how many times must $\boxed{=}$ be pressed?

S : Two times; 8 + (2 x 5) = 18.

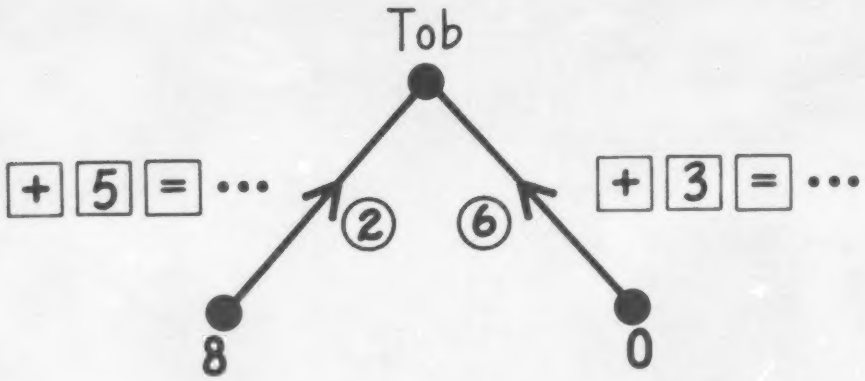
# N 1

Record this information on the board.



T: And if we start from 0 and press $\boxed{+}$ $\boxed{3}$, how many times must $\boxed{=}$ be pressed?

S: Six times; 6 x 3 = 18.



T: Which other numbers could Tob be?

S: 33.

S: 63.

S: 108.

List the numbers that Tob could be in order even though your students may not suggest them in order.

18, 33, 48, 63, 78, 93, 108, 123, 138,

After you have six to ten numbers listed, ask:

T: Do you notice anything interesting about this list of numbers?

Give the students a few minutes to study the list.

S: There is a +15 pattern; 18 + 15 = 33, 33 + 15 = 48, and so on.

T: Why do you think there is a +15 pattern?

S: 15 is a multiple of 5 and a multiple of 3.

Accept any reasonable explanation.

At this point you might ask the students to find several more numbers that Tob could be. The students should conclude that Tob could be: 18, 33, 48, 63, 78, 93, 108, 123, 138, 153, and so on.

Challenge the class to think more carefully about the numbers that Tob could be by asking questions such as:

T: What is the largest number smaller than 200 that Tob could be?

S: 198.

T: What is the smallest number larger than 200 that Tob could be?

S: 213.

**T:** What is the largest number smaller than 1,000 that Tob could be?

**S:** 993.

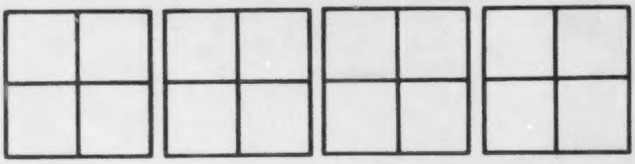
**T:** What is the smallest number larger than 1,000 that Tob could be?

**S:** 1,008.

### Second clue

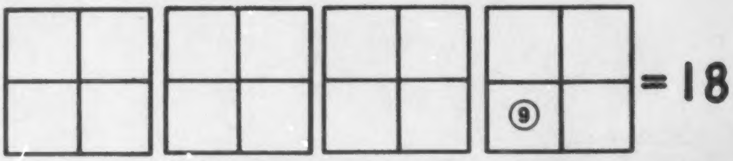
Display four Minicomputer boards and exactly one each of these checkers:

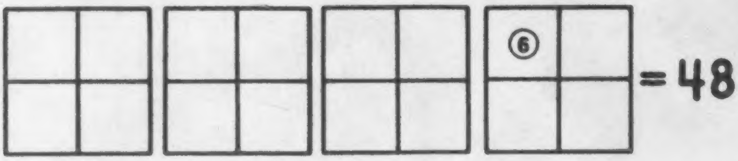
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



**T:** The second clue is that Tob can be put on this Minicomputer using exactly one of these checkers: ② through ⑨.

Ask a student to place a checker on the Minicomputer while the rest of the class tells you the number on the Minicomputer. The students should conclude that Tob could be 18 or 48.



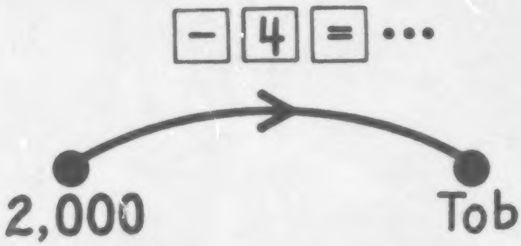


T: Could Tob be any number larger than 48?

S: No, Tob's last digit is 3 or 8. Any number, larger than 72, that can be put on the Minicomputer using exactly one of the checkers ② through ⑨ must end in 0.

**Third clue**

Draw this arrow picture on the board.



T: If you put 2,000 on the display of a hand-calculator and press $\boxed{-}$ $\boxed{4}$ and $\boxed{=}$ as many times as you like, Tob will appear on the display. Who is Tob?

Give the students a few minutes to study this situation.

S: Tob is 48. Since 2,000 is a multiple of 4, Tob must be a multiple of 4; 18 is not a multiple of 4 but 48 is.

S: I knew 0 could be displayed so I counted by fours from 0. I skipped over 18 and landed on 48; so Tob is 48.

The class should conclude that Tob is 48.

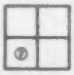
Worksheets N01 and N02 are available for independent work.

Name ____________________

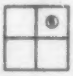
**N 01 ***

Put each number on the Minicomputer using exactly one of these checkers:

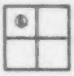
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



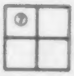
= 14



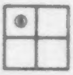
= 24



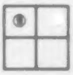
= 40



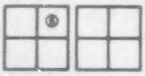
= 56



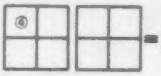
= 48



= 72



= 200



= 320

(Other solutions are possible.)

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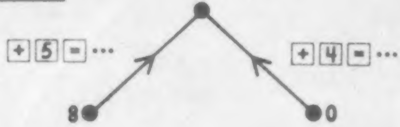
Name ____________________

**N 02 ****

Pit is a secret number.

**First clue**



Pit could be 20, 40, 60, 80, 100, 120, 140, 160, and so on.

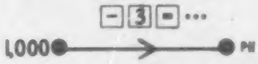
**Second clue**

Pit can be put on the Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

Pit is 20 or 40.

**Third clue**



Pit is 20.

## 1.2 Who are Flip and Flop?

CAPSULE LESSON SUMMARY

Use a string picture to review the relation $=_3$ and observe that two numbers are equal modulo 3 if and only if their difference is a multiple of 3. Solve a detective story in which clues involve writing numbers by inserting parentheses into the expressions "6 + 4 × 5 - 3" and "8 × 4 + 6 ÷ 2", the relation "is a positive divisor of", and the relation $=_3$.

MATERIALS

Teacher: Colored chalk

Student: Paper; pencil; Worksheets N03, N04, N05, and N06

DESCRIPTION OF LESSON

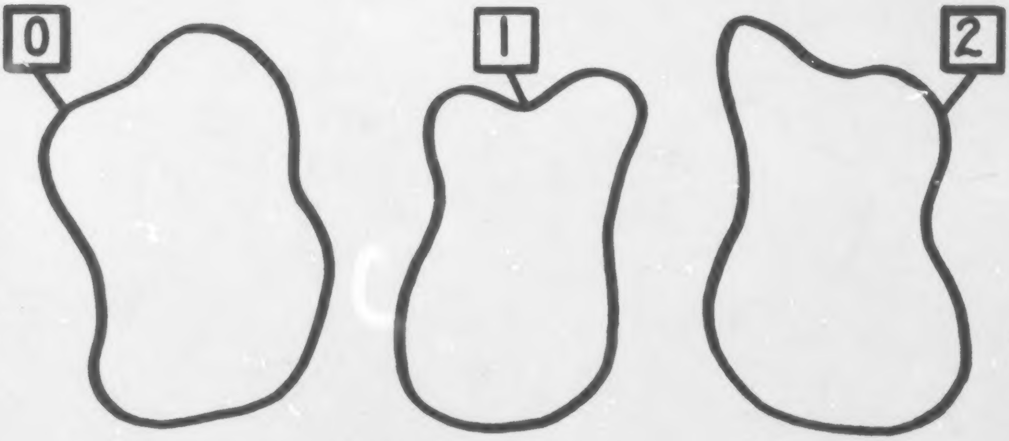
Exercise 1

Write these number stories on the board.

$$25 =_3 4$$
$$20 =_3 30$$

T: Do you remember the relation $=_3$ (read: equal modulo three)?

Let the students recall as much as they can about this relation. Draw three strings on the board.



# N3

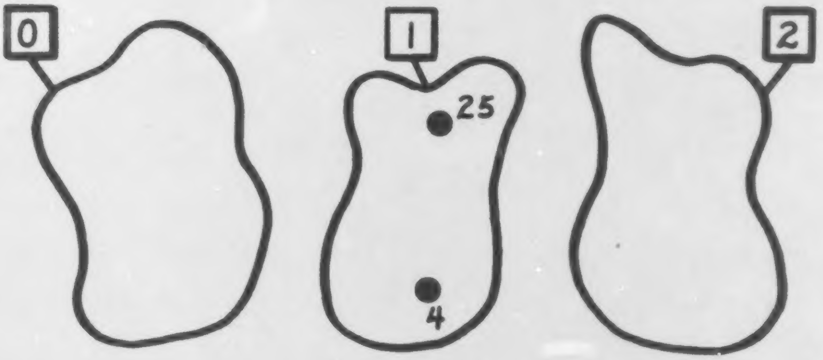
**T:** What do you think these strings are for? What do they have to do with the relation $=_3$?

**S:** The 0-string is for multiples of 3, the 1-string is for numbers which are 1 more than a multiple of 3, and the 2-string is for numbers which are 2 more than a multiple of 3. Two numbers are equal modulo 3 if and only if they are in the same string.

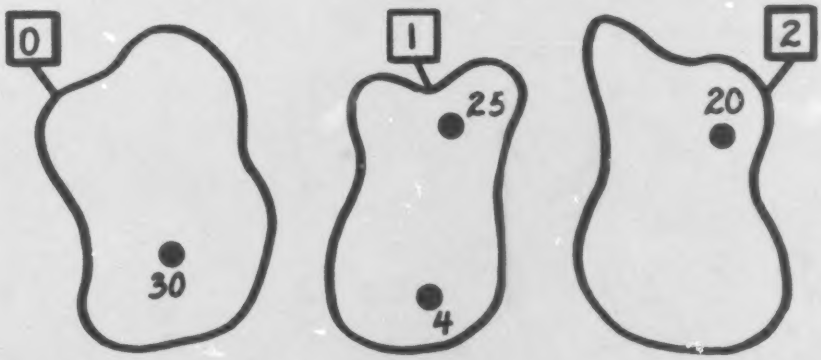
If an explanation of this relation is not forthcoming, give the explanation yourself.

**T:** Good. Are the number stories on the board true or false?

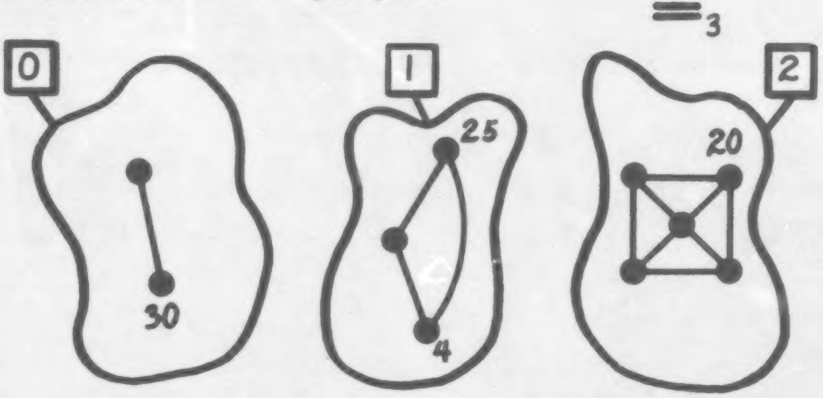
**S:** 25 is equal modulo 3 to 4, because 25 and 4 are both 1 more than a multiple of 3. They both are in the 1-string.



**S:** 20 is not equal modulo 3 to 30. 20 is in the 2-string and 30 is in the 0-string.

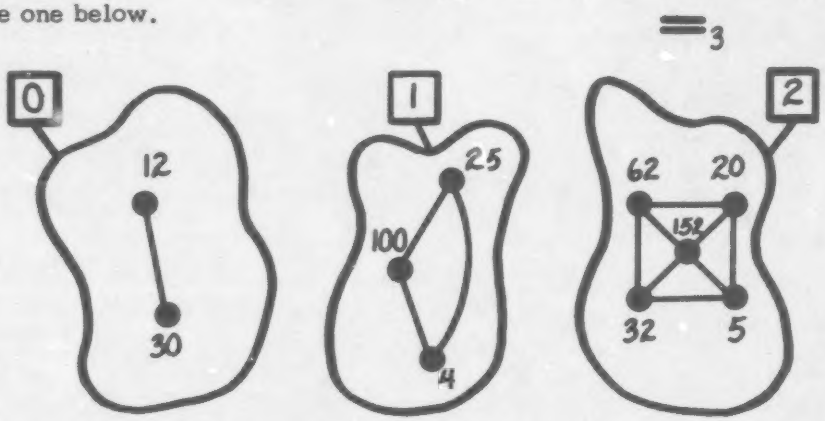


Add these dots and red cords to your picture.


=3
0
1
2
30
25
4
20

T: The red cords are for $=_3$. Let's label the dots. Many solutions are possible.

Invite students to label the dots. After a few minutes your picture might look like the one below.


=3
0
1
2
12
30
25
100
4
62
20
152
32
5

T: Give me some pairs of numbers that are not equal modulo 3.

S: 15 and 16.

Write the students' suggestions on the board.

$$15 \neq_3 16$$

$$20 \neq_3 30$$

$$102 \neq_3 200$$

$$3 \neq_3 13$$

## Exercise 2

Tell the class that they are going to solve a detective story about Flip and Flop.

**First clue**

Write this information on the board.

$$\text{Flip: } 6 + 4 \times 5 - 3$$

$$\text{Flop: } 8 \times 4 + 6 \div 2$$

T: Flip's and Flop's signatures are written on the board but they forgot to put in the parentheses. On your paper, find which numbers Flip and Flop could be.

Let the students work independently on this problem for a while. When most students have finished, ask several students to write the names for Flip and Flop on the board. A complete list of numbers that Flip and Flop could be is given below and on the next page.

$$\text{Flip: } (6 + 4) \times (5 - 3) = 20$$

$$((6 + 4) \times 5) - 3 = 47$$

$$(6 + (4 \times 5)) - 3 = 23$$

$$6 + ((4 \times 5) - 3) = 23$$

$$6 + (4 \times (5 - 3)) = 14$$

$$\text{Flop: } (8 \times 4) + (6 \div 2) = 35$$

$$((8 \times 4) + 6) \div 2 = 19$$

$$(8 \times (4 + 6)) \div 2 = 40$$

$$8 \times ((4 + 6) \div 2) = 40$$

$$8 \times (4 + (6 \div 2)) = 56$$

The students should conclude that Flip could be 20, 47, 23, or 14 and that Flop could be 35, 19, 40, or 56.

### Second clue

T: Flip is a positive divisor of Flop. On your paper, write possible couples of numbers for Flip and Flop.

S: Flip could be 20 and Flop could be 40, because 20 is a positive divisor of 40.

Write "(20, 40)" on the board.

S: 14 is a positive divisor of 56, so Flip could be 14 and Flop could be 56.

Write "(14, 56)" on the board.

S: These are the only pairs of numbers that Flip and Flop could be. Flip could be 23 or 47, but neither 23 nor 47 is a positive divisor of any number that Flop could be.

The students should conclude that (Flip, Flop) could be (20, 40) or (14, 56).

# N 3

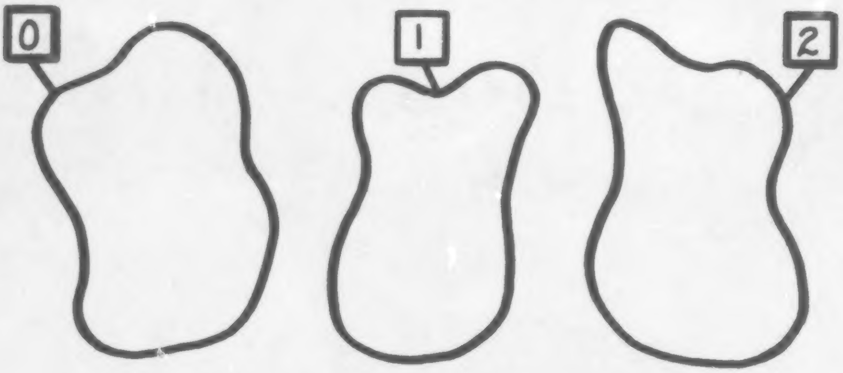
**Third clue**

Write this number story on the board.

$$\text{Flip} =_3 \text{Flop}$$

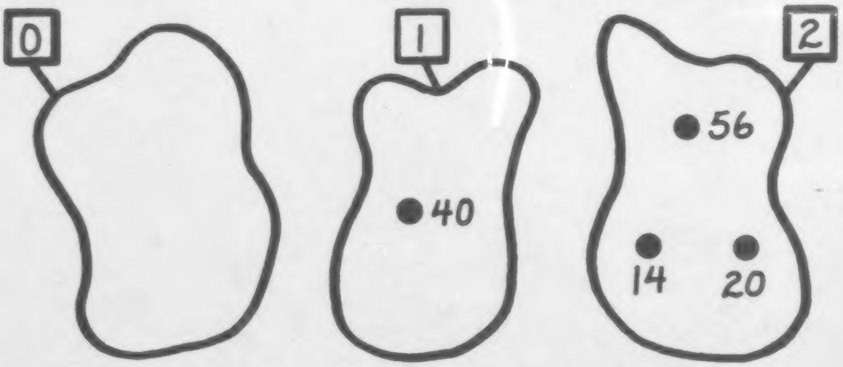
T: The third clue is given by this number story.

Draw these strings on the board as a reminder.



T: Who are Flip and Flop? Write their names on your paper.

Give the students a minute or two to think about this problem. Then invite students to put 20, 40, 14, and 56 in the string picture.



The students should conclude that Flip is 14 and Flop is 56.

Worksheets N03, N04, N05, and N06 are available for independent work.

Name ____________________ **N 03 ***

Complete these number stories.

$$(8 \times 6) + (4 \div 2) = \underline{50}$$

$$(8 \times (6 + 4)) \div 2 = \underline{40}$$

$$((8 \times 6) + 4) \div 2 = \underline{26}$$

$$8 \times ((6 + 4) \div 2) = \underline{40}$$

$$8 \times (6 + (4 \div 2)) = \underline{64}$$

---

Name ____________________ **N 04 ****

First clue

Rig forgot to put the parentheses in his signature:

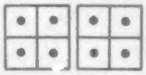
$$2 \times 3 + 4 \times 5$$

Show all the possible ways to put parentheses in this expression and find the numbers that Rig could be.

$$2 \times (3 + (4 \times 5)) = 46$$
$$2 \times ((3 + 4) \times 5) = 70$$
$$((2 \times 3) + 4) \times 5 = 50$$
$$(2 \times (3 + 4)) \times 5 = 70$$
$$(2 \times 3) + (4 \times 5) = 26$$

Rig could be 46, 70, 50, or 26.

Second clue



By moving exactly two of these checkers to other squares, you will find Rig.
Rig could be 70 or 50.

Third clue

Rig is not a positive divisor of 200.

Rig is 70.

---

Name ____________________ **N 05 *****

Pim and Pam are different secret numbers.

First clue

Pim and Pam both forget to put the parentheses in their signatures.

Pim's signature: $6 \times 8 + 4 \div 2$

Pam's signature: $6 + 8 \times 4 - 2$

Pim could be 26, 36, 50, or 60.
Pam could be 22, 28, 36, or 54.

Second clue

$$\text{Pim} =_3 \text{Pam}$$

(Pim, Pam) could be (36, 54) or (60, 54) or (60, 36).

Third clue

5 is not a divisor of Pim.

Pim is 36. Pam is 54.

---

Name ____________________ **N 06 ******

First clue

Crack forgot to put the parentheses in his signature.

$$6 + 5 \times 4 - 3 \times 2$$

On a separate sheet of paper, find all the possible ways to put parentheses in the expression.

Crack could be $^{-}22$, $^{-}4$, 16, 20, 22, 38, 40, 46, 82.

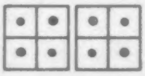
(Hint: There are between seven and fourteen possibilities for Crack.)

Second clue

$$\text{Crack} =_3 100$$

Crack could be 16, 22, 40, 46, or 82.

Third clue



By moving exactly one of these checkers to another square, you will find Crack.

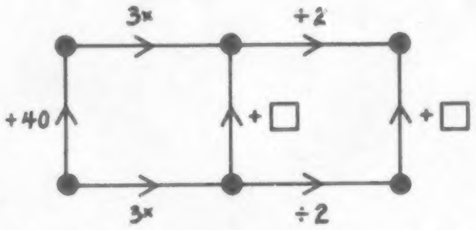
Crack is 46.

# N 8

## 1.3 Composition of Functions

CAPSULE LESSON SUMMARY

Label the arrows in an arrow picture involving arrows for multiplication functions. Using composition of addition and multiplication functions and of addition and division functions, label arrows in a picture such as:



With the hand-calculator as support, label arrows which complete a cycle of addition, multiplication, and division arrows.

MATERIALS

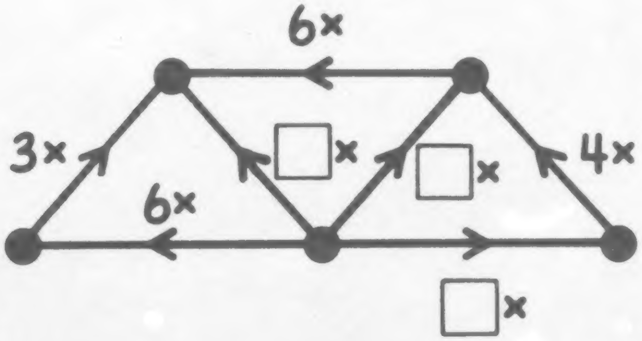
Teacher: Colored chalk

Student: Hand-calculator; colored pencils; Worksheets N07, N08, N09, and N10.

DESCRIPTION OF LESSON

### Exercise 1

Draw this part of an arrow picture on the board.

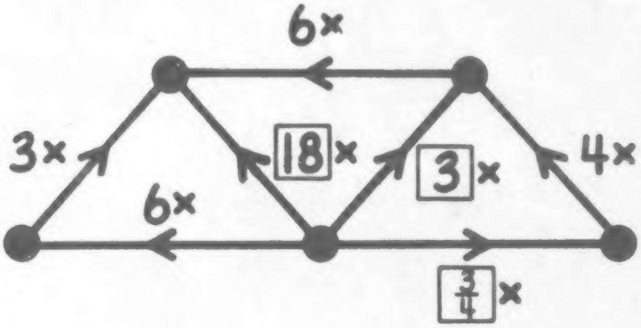


Invite students to fill in the boxes for the arrows. With the given information, they should start by labeling the left-most "unknown" arrow and work in a clockwise direction. Encourage students to trace the appropriate arrows as they give their explanations.

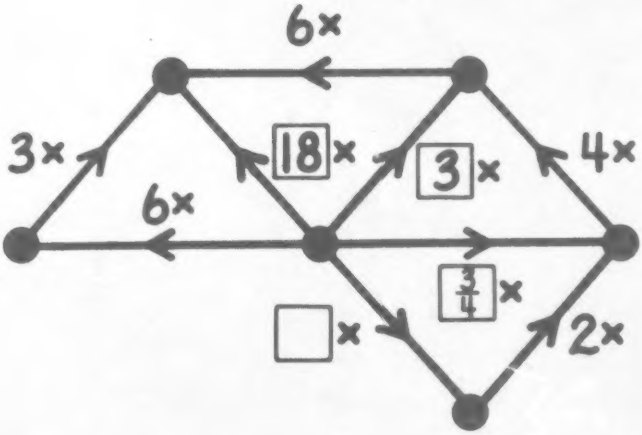
S: 6 x followed by 3 x is 18 x.

S: 18 x followed by "against" 6 x is $\frac{18}{6}$ x or 3 x.

S: 3 x followed by "against" 4 x is $\frac{3}{4}$ x.



Add two more arrows to your picture as shown below and ask the class to fill in the box.



# N 8

For the benefit of students who have difficulty labeling this arrow, trace a return arrow for 2 x.

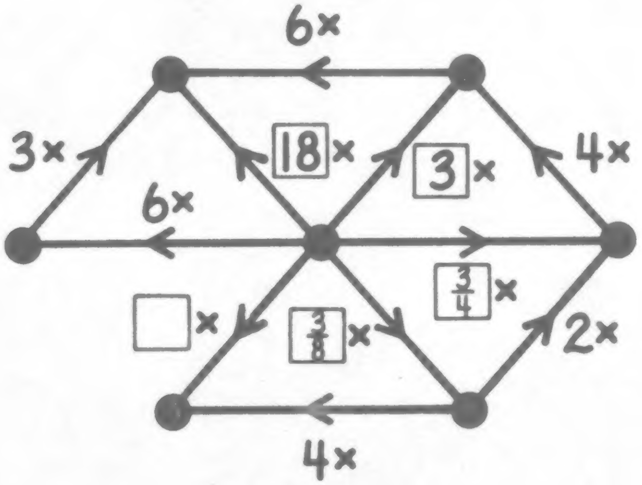
T: What could this arrow be for?

S: $\frac{1}{2}x$ (or ÷ 2).

T (tracing the appropriate arrows): $\frac{3}{4}x$ followed by $\frac{1}{2}x$. What could the "unknown" red arrow by for?

S: $\frac{3}{4}x$ followed by $\frac{1}{2}x$ is $\frac{3}{8}x$, because $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$.

Label the arrow "$\frac{3}{8}x$" and add two more arrows to your picture.



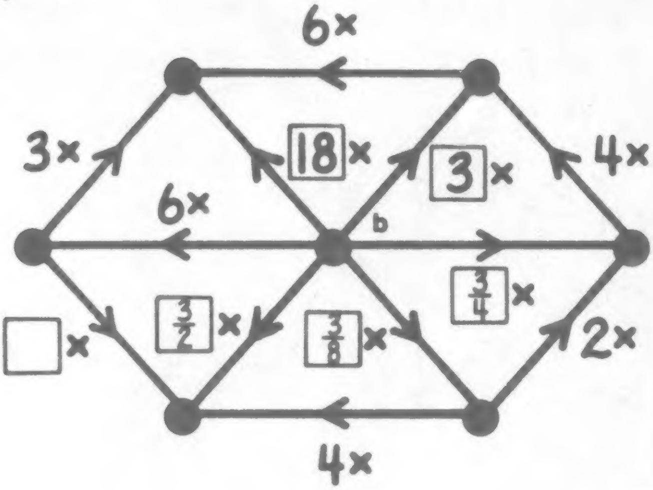
Invite a student to explain how to fill in the box.

S: $\frac{3}{8}x$ followed by 4 x is $\frac{12}{8}x$, because $\frac{3}{8} \times 4 = \frac{12}{8}$.

T: Can you give my another name for the function $\frac{12}{8}x$?

S: $\frac{3}{2}x$ or $\frac{6}{4}x$ or $\frac{24}{16}x$, and so on.

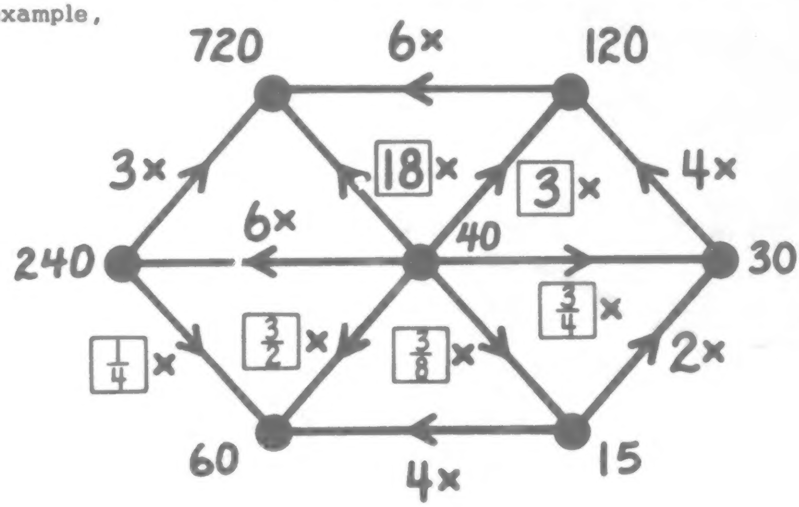
Label the arrow "$\frac{3}{2}$x" and draw the arrow shown below. Invite a student to fill in the box.


6x
3x
18 x
3 x
4x
6x
b
$\frac{3}{2}$ x
$\frac{3}{8}$ x
$\frac{3}{4}$ x
x
2x
4x

Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

S: "Against" 6 x followed by $\frac{3}{2}$ x is $\frac{1}{4}$ x, because $\frac{1}{6} \times \frac{3}{2} = \frac{3}{12} = \frac{1}{4}$.

Label the dot at b "40" and ask students to label the other dots. Each time a dot is labeled, ask the class to check that the arrows read correctly. For example,


720
6x
120
3x
18 x
3 x
4x
6x
40
240
30
$\frac{1}{4}$ x
$\frac{3}{2}$ x
$\frac{3}{8}$ x
$\frac{3}{4}$ x
2x
60
4x
15

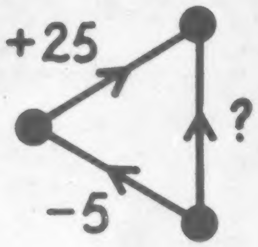
T (tracing the $\frac{3}{4}$ x arrow): Calculate "$\frac{3}{4} \times 40$".

S: $40 \div 4 = 10$ and $3 \times 10 = 30$, so $\frac{3}{4} \times 40 = 30$.

# N 8

## Exercise 2

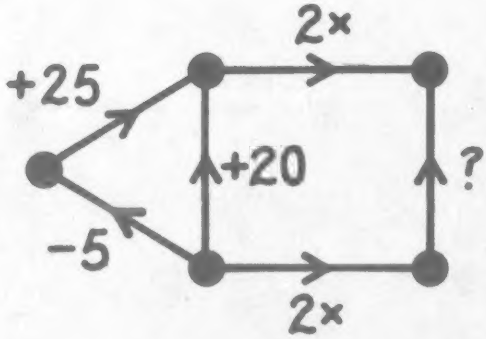
Draw this picture on the board.



T:  What could this arrow be for?

S:  + 20; - 5 followed by + 25 is + 20.

Label the arrow "+ 20" and extend your picture.

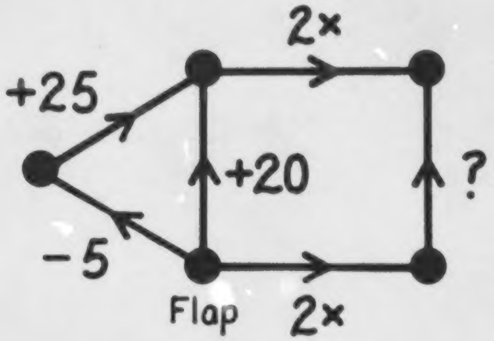


T:  What could this arrow be for?

Give the class a couple minutes to study this situation.

S: +40.

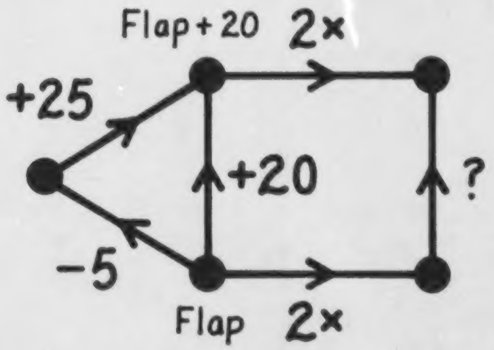
Check the student's response in the following way. Label the lower left dot "Flap".


2×
+25
+20
?
−5
Flap
2×

T (pointing to "Flap"): Suppose we put some number here. Let's call it "Flap". (Trace the + 20 arrow to its ending dot.) What can we say about the number here?

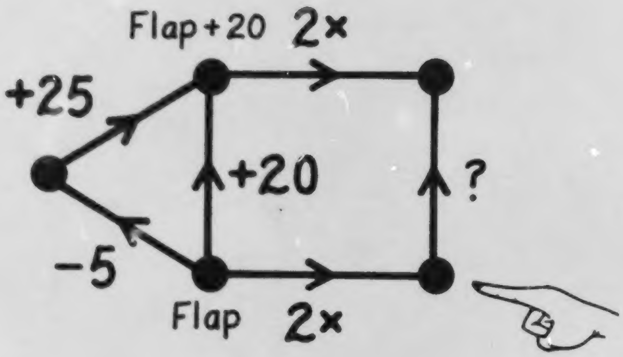
S: It is 20 more than Flap.

Label the dot "Flap + 20".


Flap + 20
2×
+25
+20
?
−5
Flap
2×

## N 8

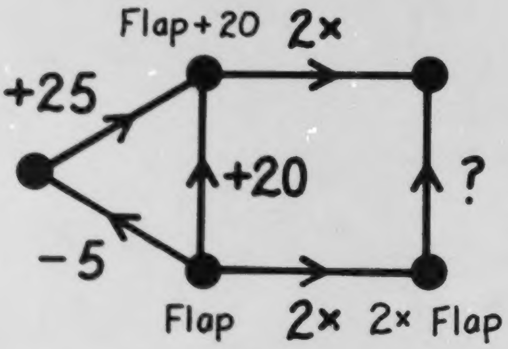
Point to the lower right dot.


Flap + 20
2x
+25
+20
?
-5
Flap
2x

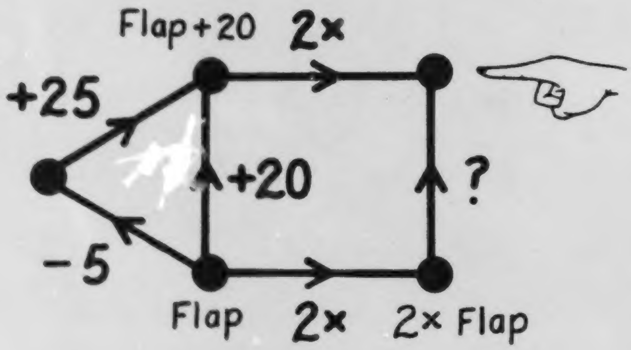
T: What do we know about this number?

S: It is two times Flap.

Label the dot "2 x Flap".

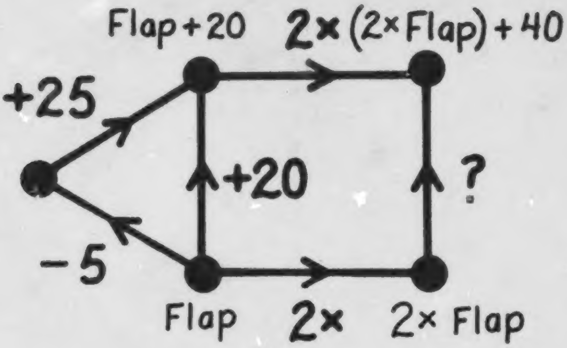

Flap + 20
2x
+25
+20
?
-5
Flap
2x
2x Flap

Point to the upper right dot.


Flap + 20
2x
+25
+20
?
-5
Flap
2x
2x Flap

T: What about this number?

S: It is 2 x (Flap + 20) or (Flap + 20) + (Flap + 20) or (2 x Flap) + 40.

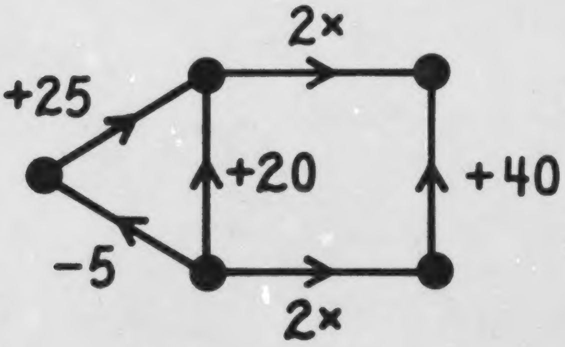

Flap+20
2×(2×Flap)+40
+25
+20
?
−5
Flap
2×
2× Flap

Trace the arrow from 2 x Flap to (2 x Flap) + 40.

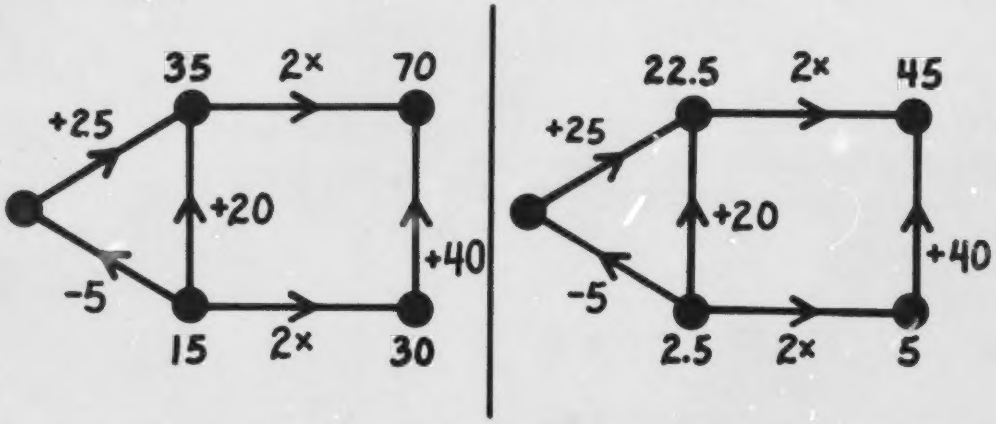
T: What could this arrow be for?

S: +40.

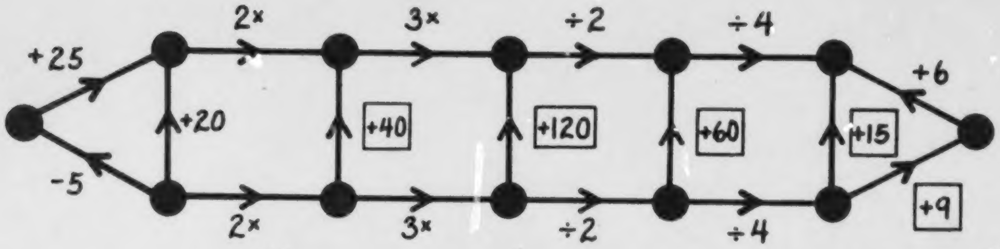
Label the arrow "+ 40" and then erase the labels for the dots.



Illustrate the "+40" relationship by choosing several different numbers for Flap and then labeling the other dots accordingly. For example,

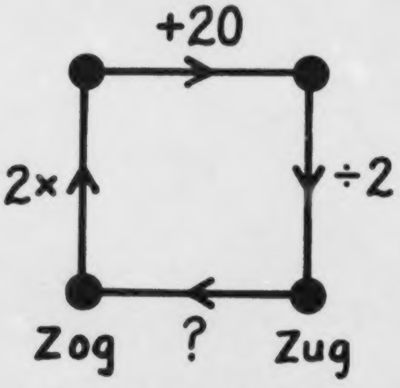

35
2×
70
+25
+20
+40
-5
15
2×
30
22.5
2×
45
+25
+20
+40
-5
2.5
2×
5

Continue this activity by extending the arrow picture and asking students to provide the labels shown in boxes below.


+25
2×
3×
÷2
÷4
+6
+20
+40
+120
+60
+15
-5
2×
3×
÷2
÷4
+9

Exercise 3

Draw this arrow picture on the board and provide a hand-calculator for each student.


+20
2×
÷2
Zog
?
Zug

100

T: Choose any number you wish for Zog and display the number on your calculator. Hide the display and then push the appropriate keys to follow the arrow road until you come to Zug. Next, come back to Zog with just one operation. (Trace the arrow labeled "?".) What could this arrow be for?

Give the students a few minutes to experiment and to solve this problem.

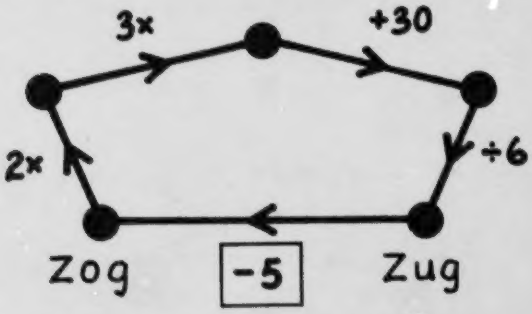
S: -10.

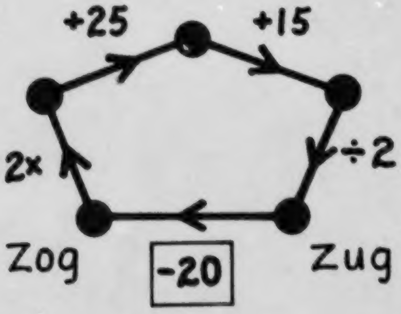
T: How do you know?

S: I followed the arrow road; 2 x followed by + 20 followed by ÷ 2. I know that Zug will be 10 more than Zog, so the arrow from Zug to Zog could be for - 10.

T: Check the display to see if you are correct.

Repeat this activity using the arrow roads shown below.



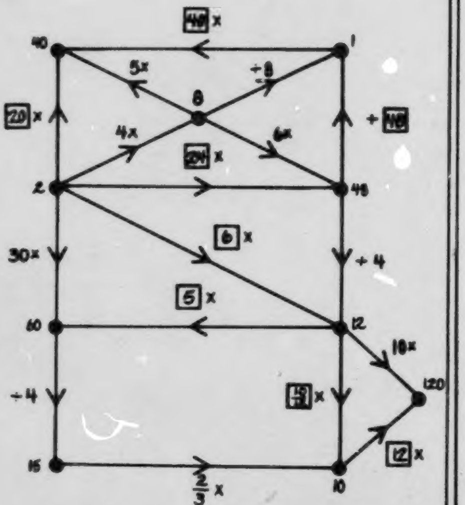


Worksheets N07, N08, N09, and N10 are available for independent work.

Name

N 07 *

Label the dots and fill in the boxes for the arrows.

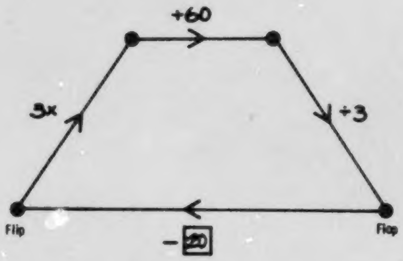


$\boxed{40}$ x
40
1
5x
÷8
8
$\boxed{20}$ x
÷ $\boxed{48}$
4x
6x
$\boxed{24}$ x
2
48
$\boxed{6}$ x
30x
÷ 4
$\boxed{5}$ x
60
12
10x
120
÷4
$\boxed{\frac{10}{12}}$ x
$\boxed{12}$ x
15
10
$\frac{2}{3}$ x

(Different names for $\frac{10}{12}$ could be given; e.g., $\frac{10}{12} = \frac{5}{6} = \frac{40}{48} = \frac{50}{60} = \ldots .$)

Name

N 08 **



+60
3x
÷3
Flip
Flop
$-\boxed{20}$

Complete these charts and fill in the box for the arrow from Flop to Flip.

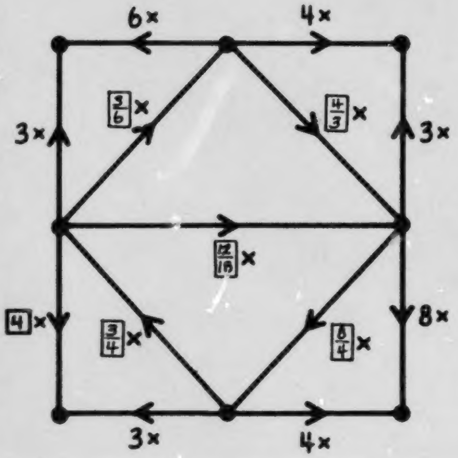
| Flip | Flop |
|---|---|
| 2 | 22 |
| 5 | 25 |
| 10 | 30 |

| Flip | Flop |
|---|---|
| 0.2 | 20.2 |
| 1.5 | 21.5 |
| 10.5 | 30.5 |

Name

N 09 ***

Fill in the boxes for the arrows.



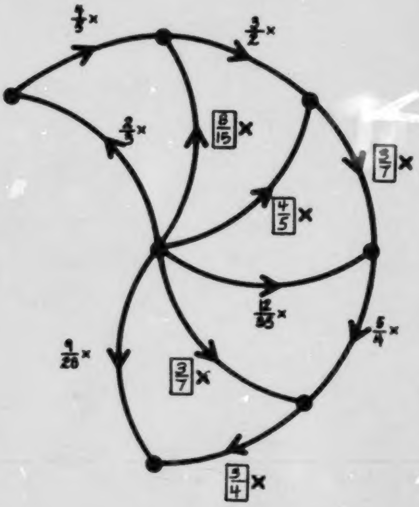
6x
4x
$\boxed{\frac{3}{6}}$x
$\boxed{\frac{4}{3}}$x
3x
3x
$\boxed{\frac{12}{18}}$x
$\boxed{4}$x
$\boxed{\frac{3}{4}}$x
$\boxed{\frac{8}{4}}$x
8x
3x
4x

(Other names could be used to label the arrows; e.g., $\frac{3}{6} = \frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{5}{10} = \ldots .$)

Name

N 10 ****

Fill in the boxes for the arrows.



$\frac{4}{3}$x
$\frac{3}{2}$x
$\frac{2}{3}$x
$\boxed{\frac{8}{15}}$x
$\boxed{\frac{5}{7}}$x
$\boxed{\frac{4}{5}}$x
$\frac{12}{35}$x
$\frac{5}{4}$x
$\frac{9}{28}$x
$\boxed{\frac{3}{7}}$x
$\boxed{\frac{3}{4}}$x

(Other names could be used to label the arrows; e.g., $\frac{8}{15} = \frac{16}{30} = \frac{24}{45} = \frac{32}{60} = \ldots .$)

# 2

# RATIONAL NUMBERS[*]

In this chapter, a variety of situations provide opportunities to further explore the rational numbers and to begin developing algorithms for operating with rational numbers. The situations involve arrow diagrams, Guess My Rule games with rational numbers, the number line, string pictures, patterns in sets of calculations, and geometric partitioning of regions. Some of the many properties being reinforced by the activities in the lessons include:

- $\frac{4}{5}x$ is the composite function $4\times$ followed by $\div 5$ (or "against" $5\times$) and the converse of $\frac{4}{5}x$ is $\frac{5}{4}x$ or $\div\frac{4}{5}$.
- the composition of $\frac{2}{3}x$ followed by $\frac{5}{7}x$ is $\frac{10}{21}x$, and $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$.
- $\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}$.
- a rational number may have many fractional names and, if a decimal, many decimal names.
- $\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$ if and only if $\frac{a}{c} = \frac{b}{d}$.
- addition of rational numbers can often be performed by first using fractions and then using decimal names or vice versa; e.g., $\frac{2}{5} + \frac{3}{2} = \frac{4}{10} + \frac{15}{10} = \frac{19}{10} = 1.9 = 0.4 + 1.5$.

---

[*] The rational numbers include integers and all numbers which have fractional names with integer numerators and positive integer denominators; e.g., $\frac{3}{4}$, $\frac{81}{732}$, $\frac{9}{5}$, $\frac{^-1}{2}$, $\frac{^-7}{3}$. Decimal numbers are also rational numbers since, for example, $0.75 = \frac{3}{4}$, $1.3 = \frac{13}{10}$, and so on. In communicating with students, we suggest you ignore the distinction between rational numbers and their fractional names and refer to them simply as "fractions". The lesson descriptions use "fraction" to refer to a certain kind of name for a rational number, but we feel it is better pedagogical practice at this level not to force this distinction in student discussions, and feel it is unnecessary to introduce the term "rational number" with the students.

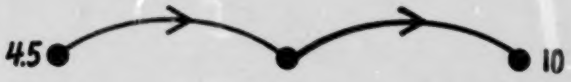
## CAPSULE LESSON SUMMARIES

Section 2.1: Solve several sets of problems involving addition, multiplication, and division of decimal numbers. Observe some patterns, use the idea of money, and look at related problems with fractions to support the calculations with decimals. Label parts of the number line where the graduations are in the hundredths and thousandths.
(Worksheets N11, N12, N13, N14)

Section 2.2: Use composition of multiplication functions to label some arrows in given arrow pictures. Observe that for a $\frac{5}{6}\times$ arrow the return arrow is for $\frac{6}{5}\times$. Find that $\frac{2}{3}\times$ followed by $\frac{4}{5}\times$ followed by $\frac{6}{7}\times$ is $\frac{48}{105}\times$ and relate this to the calculation "$\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}$". Find decimal names for some fractions and use fractions to help solve multiplication problems with decimals.
(Worksheets N15, N16, N17, N18)

Section 2.3: Do addition calculations such as "$\frac{1}{5} + \frac{1}{4}$" and "$\frac{3}{4} + \frac{1}{6}$" first by using appropriate partitions of regions. Then do such calculations only thinking about the partitions, but choosing appropriate (useful) fractional names for the numbers being added.
(Worksheets N19, N20, N21)

Section 2.4: Solve several sets of problems involving multiplication and division of decimal numbers. Observe some patterns, use the idea of money, and look at related problems with fractions to support the calculations with decimals. Find pairs of functions that the blue and red arrows could be for in situations such as:



(Worksheets N22, N23, N24, N25)

Section 2.5: Play a Guess My Rule game with a secret operation * where $a * b = (a \times b) + 1$. Locate decimals on the number line when non-standard names such as "$\frac{1}{2} \times 0.5$", "$0.5 \times 0.3$", and "$2 - 0.95$" are given for the numbers. In the same way, place decimals in a string picture with one string labeled "larger than 0.2".

(Worksheets N26, N27, N28)

Section 2.6: Label the dots in an arrow picture where the arrows are all for multiplication by decimals. Use the picture and appropriate return arrows to do some division calculations with decimals. Label dots in an arrow picture where the arrows are for +, -, x, or ÷ decimals. Introduce the idea of building "one-digit" arrow roads, that is, roads where the arrows must be for +, -, x, or ÷ some one digit positive whole number and build some such arrow roads.

Section 2.7: Find many fractional names for numbers such as $\frac{3}{5}$ and $\frac{5}{2}$. Play a Guess My Rule game where the rule involves a square configuration of numbers.

| | |
|---|---|
| a | b |
| c | d |

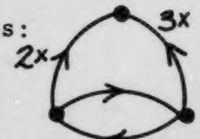
$$a \times d = b \times c \text{ or } \frac{a}{c} = \frac{b}{d} \text{ or } \frac{a}{b} = \frac{c}{d}$$

Use an arrow picture to look at the relationship between the numbers in such a configuration.

(Worksheets N29, N30, N31)

Section 2.8: Locate rational numbers in a string picture with one string labeled "larger than $\frac{1}{2}$" when the numbers are given with non-standard names such as "$\frac{2}{3} \times \frac{3}{2}$", "$\frac{3}{8} + \frac{1}{4}$", and "$2 - \frac{5}{4}$". Play a game between two teams where the object is to label the dots in a two-string picture using a set of non-standard names for the numbers.

(Worksheets N32, N33, N34)



Section 2.9 : Label arrows in an arrow picture such as: Observe that the red arrows are return arrows of each other and find two ways to label each, where one way is of the form "÷☐". Use such pictures to help do calculations involving division by rational numbers (expressed as fractions); for example,

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}.$$

In the context of an experiment of randomly picking a piece of paper, marked with a numeral or an operation symbol, from each of three boxes, do some multiplication, division, and addition calculations with fractions. Determine the probability of getting a number from $\frac{1}{3}$ to $\frac{5}{4}$ (inclusive).
(Worksheet N35)

## 2.1 Decimals #1

CAPSULE LESSON SUMMARY

Solve several sets of problems involving addition, multiplication, and division of decimal numbers. Observe some patterns, use the idea of money, and look at related problems with fractions to support the calculations with decimals. Label parts of the number line where the graduations are in the hundredths and thousandths.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil; Worksheets N11, N12, N13, and N14

DESCRIPTION OF LESSON

Exercise 1

Write these problems on the board.

$$4.3 + \boxed{\phantom{5.638}} = 10$$
$$4.36 + \boxed{\phantom{5.638}} = 10$$
$$4.362 + \boxed{\phantom{5.638}} = 10$$

T: Copy these problems and fill in the boxes.

Let the students work independently for a few minutes before checking the problems collectively. Answers are in the boxes below.

$$4.3 + \boxed{5.7} = 10$$
$$4.36 + \boxed{5.64} = 10$$
$$4.362 + \boxed{5.638} = 10$$

# N 2

**T:** Did anyone find what you think is a simple way to solve these problems?

Let the students express their ideas.

**S** (pointing to the second problem): Start from the right with "6". There must be a "4" in that place because 6 + 4 = 10. For the other two digits, add a number to get 9, so a "6" (3 + 6 = 9) and a "5" (4 + 5 = 9) are needed.

Write this set of problems on the board and ask the students to complete the calculations.

$$\begin{array}{rl} 5 \times 0.6 & = \\ 5 \times 0.06 & = \\ 5 \times 0.006 & = \\ 50 \times 0.6 & = \\ 50 \times 0.06 & = \end{array}$$

Observe the students' work and give help when needed. Encourage students who have difficulty to do the first problems thinking about money. For example, a student could think of "5 × 0.6" as five times sixty cents. Then suggest they look for patterns.

Check the problems collectively (answers are in the boxes below) and invite students to observe patterns in the problem set. One observation is suggested below.

$$\begin{tikzcd}
5 \times 0.6 \arrow[ddd, bend right, "10\times"'] & = \boxed{3} \arrow[ddd, bend left, "10\times"] \\
5 \times 0.06 & = \boxed{0.3} \\
5 \times 0.006 & = \boxed{0.03} \\
50 \times 0.6 & = \boxed{30} \\
50 \times 0.06 & = \boxed{3}
\end{tikzcd}$$

Continue this activity by asking the students to do these calculations.

$$0.5 \times 0.6 =$$
$$0.5 \times 0.06 =$$

T: How can we perform these calculations?

S: Use fractions.

If necessary, suggest this method yourself.

T: Can you give me a fraction for 0.5?

S: "$\frac{5}{10}$".

T: ... a fraction for 0.6?

S: "$\frac{6}{10}$".

Write this information on the board.

$$0.5 \times 0.6 = \frac{5}{10} \times \frac{6}{10}$$

T: What number is $\frac{5}{10} \times \frac{6}{10}$?

S: $\frac{30}{100}$.

$$0.5 \times 0.6 = \frac{5}{10} \times \frac{6}{10} = \frac{30}{100}$$

T: Can you give a decimal name for $\frac{30}{100}$?

S: "0.30" or "0.3".

$$0.5 \times 0.6 = \frac{5}{10} \times \frac{6}{10} = \frac{30}{100} = 0.30 = 0.3$$

Discuss the second problem in a similar manner. The class should conclude:

$$0.5 \times 0.06 = \frac{5}{10} \times \frac{6}{100} = \frac{30}{1{,}000} = 0.030 = 0.03$$

Write a new set of problems on the board and ask the students to complete the calculations. Answers are in the boxes.

$$54 \div 10 = \boxed{5.4}$$
$$5.4 \div 10 = \boxed{0.54}$$
$$54 \div 100 = \boxed{0.54}$$
$$5.4 \div 100 = \boxed{0.054}$$

Use money as a support with students who have difficulty. For example, suggest that a student think of "5.4 ÷ 10" as five dollars and forty cents shared equally among ten people.

Check the problems collectively. If there has been much discussion about money, some students may question 0.054.

Write "$0.004" on the board.

T: How much money is this?

S: It is less than a penny. A penny is $0.01.

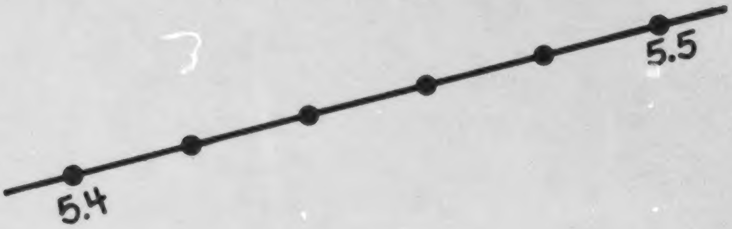
S: Then it is less than one-half of a penny, too.

T: Good. How much money is \$0.054?

S: A little less than five and one-half cents.

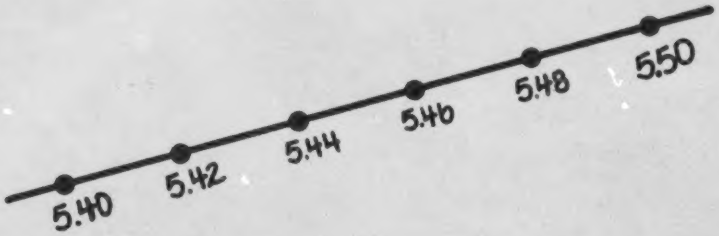
Exercise 2

Draw this part of the number line on the board.

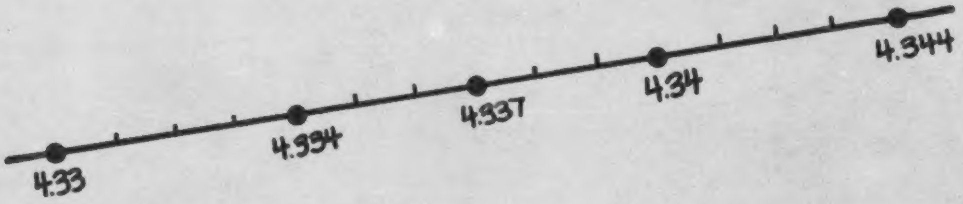


T: Copy this part of the number line and label the dots.

Allow the class to work independently. Some students may need to be reminded that 5.4 = 5.40 and 5.5 = 5.50. When most of the students have finished, call on volunteers to label the dots at the board. Answers are given below in red.



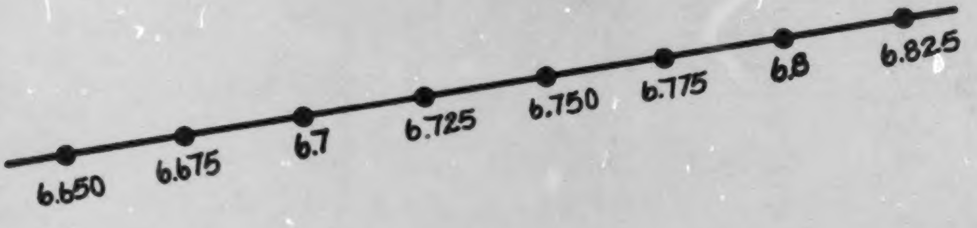
Repeat the activity using this part of the number line, or adjust the situation to the numerical level of your students. Answers are given in red.



Faster workers could work on labeling another part of the number line such as

# N 2

the one shown below while the rest of the class is finishing on the previous one. Again, answers are given in red.


6.650
6.675
6.7
6.725
6.750
6.775
6.8
6.825

Worksheets N11, N12, N13, and N14 are available.

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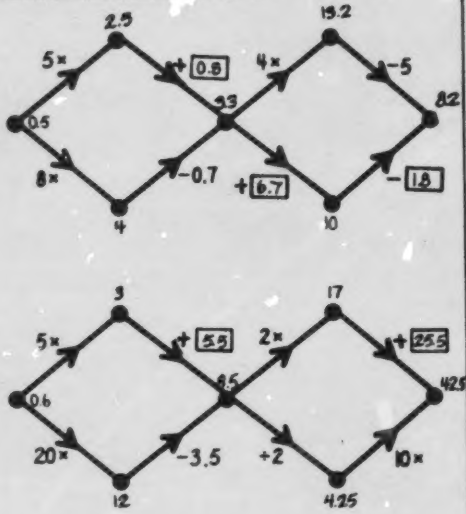
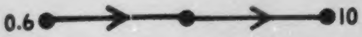
Name ____________________ **N 11 ***

Label the dots and fill in the boxes for the arrows.

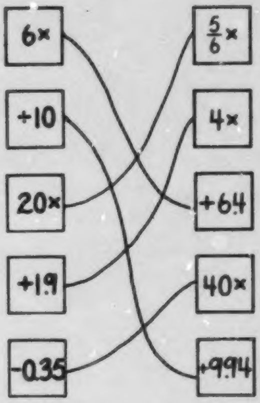

2.5
5×
+ 0.8
4×
13.2
−5
3.3
8.2
0.5
8×
−0.7
+ 6.7
− 1.8
4
10
3
5×
+ 5.5
2×
17
+ 25.5
8.5
42.5
0.6
20×
−3.5
÷2
10×
12
4.25

---

Name ____________________ **N 12 ****



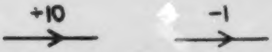
Pair the tags.


6×
÷10
20×
+1.9
−0.35
5/6 ×
4×
+6.4
40×
+9.94

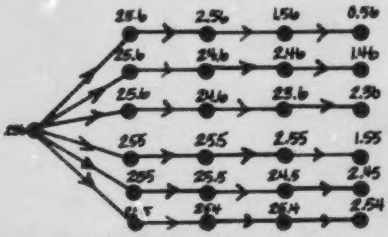
---

Name ____________________ **N 13 *****

First clue

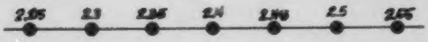


Flip is the ending number of an arrow road starting at 256 and using exactly two red arrows and two blue arrows.


256
25.6
2.56
1.56
0.56
25.6
24.6
2.46
1.46
25.6
24.6
23.6
2.36
255
25.5
2.55
1.55
255
25.5
24.5
2.45
25.5
254
25.4
2.54

Flip could be 0.56, 1.46, 2.36, 1.55, 2.45, or 2.54.

Second clue


2.25
2.3
2.35
2.4
2.45
2.5
2.55

Label each dot. Flip is one of these numbers.

Flip is 2.45.

---

Name ____________________ **N 14 ******

Put each number on a hand-calculator using just these keys:

2, 8, +, −, ×, ÷, =

Write the keys in the order you use them. You may use a key more than once.

It costs 1¢ each time you press a key. Try not to spend more than 15¢ for each number.

2.8 2 + 8 ÷ = = × 2 8 =

0.03 2 + 8 ÷ = = + = =

0.88 2 + 8 ÷ = = = × 8 8 =

8.4 2 + 8 ÷ = = × 2 = + − + 8 =

0.18 2 + 8 ÷ = = = + − = + = = + =

(Many solutions are possible. Refer to Section 2(f) of the General Introduction for a discussion of hand-calculator characteristics which could affect the solutions.)

# N 6

## 2.2 Multiplication of Rational Numbers

CAPSULE LESSON SUMMARY

Use composition of multiplication functions to label some arrows in given arrow pictures. Observe that for a $\frac{5}{6}x$ arrow the return arrow is for $\frac{6}{5}x$. Find that $\frac{2}{3}x$ followed by $\frac{4}{5}x$ followed by $\frac{6}{7}x$ is $\frac{48}{105}x$ and relate this to the calculation "$\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}$". Write decimal names for some fractions and use fractions to help solve multiplication problems with decimals.

MATERIALS

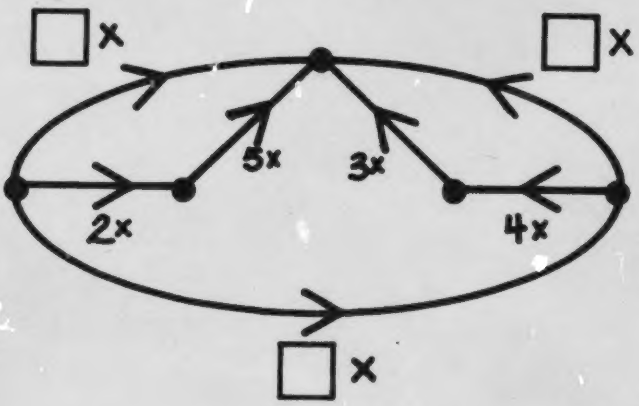
Teacher: Colored chalk

Student: Paper; colored pencils; Worksheets N15, N16, N17, and N18

DESCRIPTION OF LESSON

Exercise 1

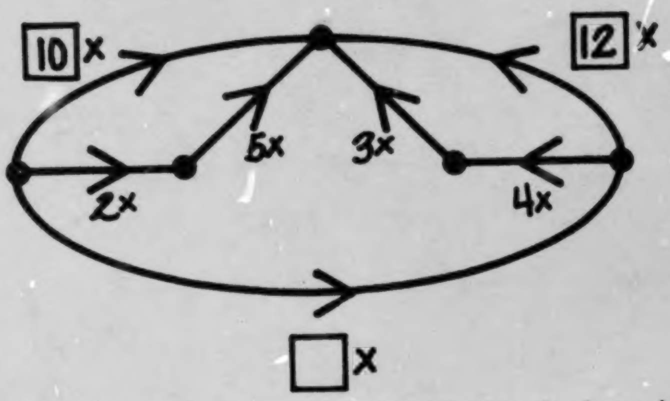
Draw this arrow picture on the board.



Invite students to fill in the boxes. Each time allow the student to choose which arrow to label.

S (tracing the appropriate arrows): 2x followed by 5x is 10x.

S (tracing the appropriate arrows): 4 x followed by 3 x is 12 x.



For the benefit of students who have difficulty labeling the lower blue arrow, trace "with" and "against" the upper blue arrows in turn as you ask:

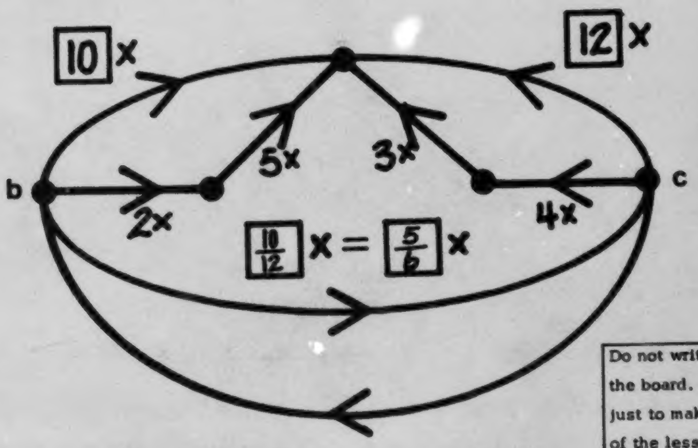
T: 10 x followed by "against" 12 x is . . . ?

S: $\frac{10}{12}$ x.

T: Yes. Can you suggest another name for $\frac{10}{12}$ x?

S: "$\frac{5}{6}$ x".

Fill in the box and add the return arrow as shown in the illustration below.

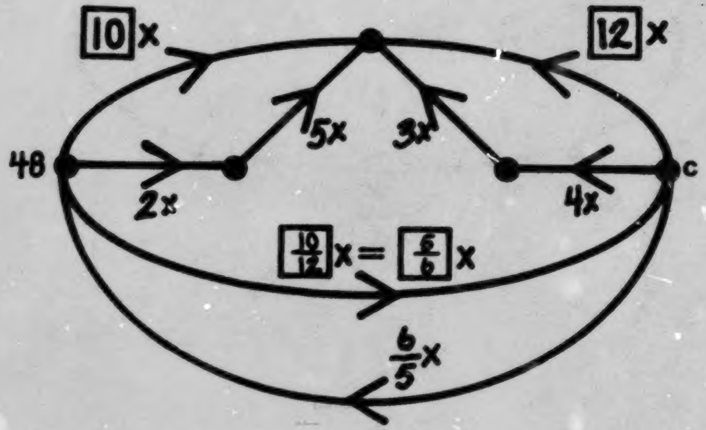


Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (tracing the return arrow from c to b): What could this return arrow be for?

S: $\frac{6}{5}$x or $\frac{12}{10}$x.

Label the return arrow "$\frac{6}{5}$x" and label b "48".



10 x
12 x
5x
3x
48
c
2x
4x
10/12 x = 5/6 x
6/5 x

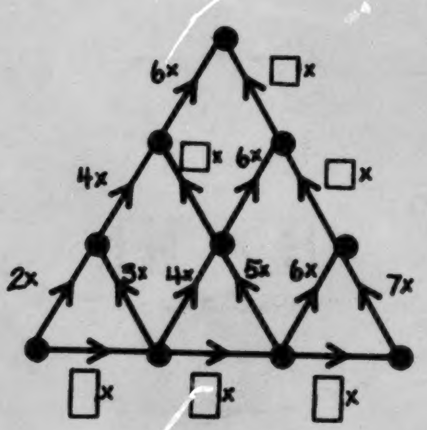
T (pointing to c): Which number is here?

S: 40, because 48 ÷ 6 = 8 and 5 x 8 = 40, so $\frac{5}{6}$ x 48 = 40.

Choose a few more numerical examples (for both b and c) appropriate to the abilities of your students to illustrate the $\frac{5}{6}$x and $\frac{6}{5}$x functions.

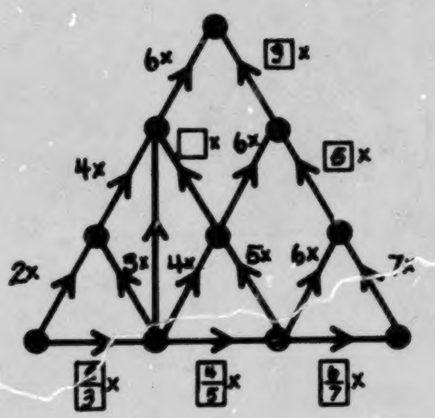
Exercise 2

Draw this arrow picture on the board and ask the students to copy it and fill in the boxes.



6x
☐x
☐x
6x
4x
☐x
2x
3x
4x
5x
6x
7x
☐x
☐x
☐x

Invite students to fill in the boxes for arrows of their choice. Answers are in the boxes below. For the benefit of students who have difficulty labeling the middle arrow, add this blue supplementary arrow to your picture.



T: What could this blue arrow be for?

S: 12 x.

T: Good. The blue arrow is for 3 x followed by 4 x or 12 x. (Point to the blank box.) What could this red arrow be for?

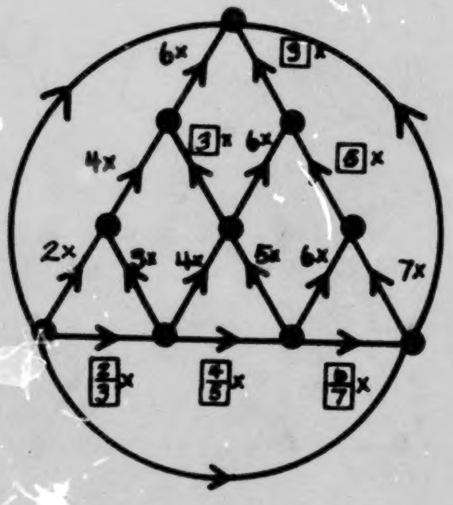
S: 3 x, because 4 x followed by 3 x is also 12 x.

Other supplementary arrows may be drawn if necessary. Erase them after their use.

Add three blue arrows to your picture (see the next illustration).

# N 6

Ask students to label them in the order suggested in the following dialogue.



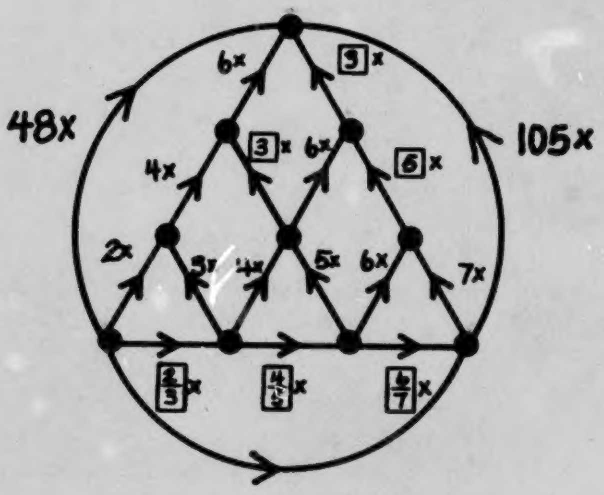
6x
3x
4x
3x
6x
6x
2x
3x
4x
5x
6x
7x
2/3 x
4/5 x
6/7 x

T: Look at the blue arrow to the left. It is for 2 x followed by 4 x followed by 6 x. What could this blue arrow be for?

S: 48 x, because 2 x 4 x 6 = 48.

T: Now look at the blue arrow to the right. It is for 7 x followed by 5 x followed by 3 x. What could this blue arrow be for?

S: 105 x, because 7 x 5 x 3 = 105.



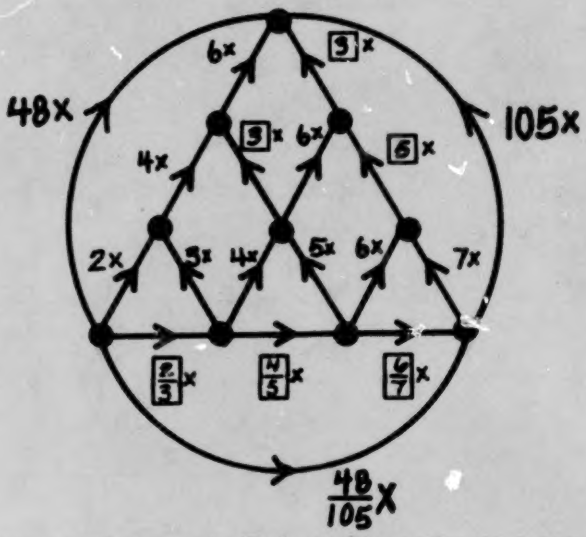
48x
105x
6x
3x
4x
3x
6x
6x
2x
3x
4x
5x
6x
7x
2/3 x
4/5 x
6/7 x

**T:** Look at the blue arrow at the bottom. What could this blue arrow be for? Use the other two blue arrows to help you.

Give the class a few minutes to study the situation.

**S:** $\frac{48}{105}$x; it is 48x followed by "against" 105x.

T (tracing "with" or "against" the appropriate arrows in turn): 48x followed by "against" 105x is $\frac{48}{105}$x.


6x
$\boxed{3}$x
48x
105x
$\boxed{3}$x
6x
$\boxed{6}$x
4x
2x
5x
4x
5x
6x
7x
$\boxed{\frac{2}{3}}$x
$\boxed{\frac{4}{5}}$x
$\boxed{\frac{6}{7}}$x
$\frac{48}{105}$x

**T:** Do you see another way to look at this arrow as a composition of arrows?

**S:** It is $\frac{2}{3}$x followed by $\frac{4}{5}$x followed by $\frac{6}{7}$x.

Write this calculation on the board.

$$\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} =$$

**T:** How do we calculate "$\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}$"?

Summarize the students comments as you complete the calculation on the board.

$$\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} = \frac{2 \times 4 \times 6}{3 \times 5 \times 7} = \frac{48}{105}$$

T: Good. Can you suggest some other names for $\frac{48}{105}$?

Accept several names for $\frac{48}{105}$ and list them on the board.

$$\frac{48}{105} = \frac{96}{210} = \frac{480}{1{,}050} = \frac{16}{35}$$

Instruct the students to copy and solve the problems below. After a few minutes, invite students to explain the calculations. Answers are in the boxes.

$$\frac{3}{5} \times \frac{15}{4} = \boxed{\frac{3 \times 15}{5 \times 4} = \frac{45}{20} = \frac{9}{4}}$$

$$\frac{4}{7} \times \frac{14}{8} = \boxed{\frac{4 \times 14}{7 \times 8} = \frac{56}{56} = 1}$$

$$\frac{3}{4} \times \frac{10}{5} = \boxed{\frac{3 \times 10}{4 \times 5} = \frac{30}{20} = \frac{3}{2}}$$

Exercise 3

Write these fractions on the board.

$$\frac{21}{10}$$

$$\frac{6}{100}$$

$$\frac{1}{5}$$

**T:** On your paper, write a decimal name for each of these fractions.

Allow a few minutes for individual work before checking the work collectively.

$$\frac{21}{10} = 2.1$$

$$\frac{6}{100} = 0.06$$

$$\frac{1}{5} = 0.2$$

Some students may have difficulty with $\frac{1}{5}$. Lead a discussion similar to the one described below.

**T:** It is easy to write decimal names for fractions when the denominator (bottom number) is 10 or 100 or 1,000, and so on. For $\frac{1}{5}$ we could find another fraction with the denominator being 10 or 100 or 1,000, and so on. Can you find such a name for $\frac{1}{5}$?

**S:** "$\frac{2}{10}$".

**T:** Good. Now write a decimal name for $\frac{1}{5}$.

$$\frac{1}{5} = \frac{2}{10} = 0.2$$

Write these fractions on the board and ask the class to write decimal names for them. Answers are in the boxes.

$$\frac{1}{4} = \boxed{\frac{25}{100} = 0.25}$$

$$\frac{6}{10} = \boxed{0.6}$$

$$\frac{125}{100} = \boxed{1.25}$$

# N 6

Write this calculation on the board.

$$0.3 \times 0.4 = \boxed{\phantom{0000}}$$

T: How can we calculate "0.3 × 0.4"?

S: We could use fractions.

If necessary, suggest this method yourself.

T: Can you give a fraction for 0.3?

S: "$\frac{3}{10}$".

T: ... a fraction for 0.4?

S: "$\frac{4}{10}$".

Write this information on the board.

$$0.3 \times 0.4 = \frac{3}{10} \times \frac{4}{10}$$

T: What number is $\frac{3}{10} \times \frac{4}{10}$?

S: $\frac{12}{100}$.

$$0.3 \times 0.4 = \frac{3}{10} \times \frac{4}{10} = \frac{12}{100}$$

T: Can you give a decimal name for $\frac{12}{100}$?

S: "0.12".

$$0.3 \times 0.4 = \frac{3}{10} \times \frac{4}{10} = \frac{12}{100} = 0.12$$

Continue this activity with the problems below. Ask the students to copy and to solve them. Check the individual work collectively by asking students to explain the calculations at the board. Answers are in the boxes.

$$2.1 \times 0.4 = \boxed{0.84}$$

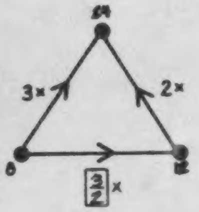
$$0.25 \times 0.5 = \boxed{0.125}$$

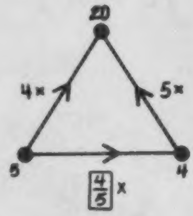
$$2.3 \times 4.2 = \boxed{9.66}$$

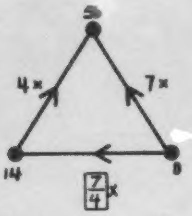
Worksheets N15, N16, N17, and N18 are available.

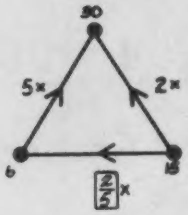
Name ______________________ **N 15** *

Label the dots and fill in the boxes for the arrows.


24
3×
2×
8
12
$\boxed{\frac{3}{2}}$×


20
4×
5×
5
4
$\boxed{\frac{4}{5}}$×

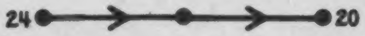

56
4×
7×
14
8
$\boxed{\frac{7}{4}}$×


30
5×
2×
6
15
$\boxed{\frac{2}{5}}$×

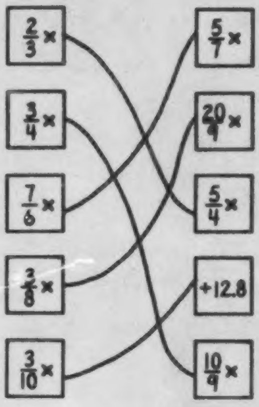
(Other names could be used to label the arrows; e.g., $\frac{3}{2} = \frac{12}{8} = \frac{6}{4} = \frac{60}{40} = \ldots .$)

---

Name ______________________ **N 16** **



Pair the tags.


$\frac{2}{3}$×
$\frac{3}{4}$×
$\frac{7}{6}$×
$\frac{3}{8}$×
$\frac{3}{10}$×
$\frac{5}{7}$×
$\frac{20}{9}$×
$\frac{5}{4}$×
+12.8
$\frac{10}{9}$×

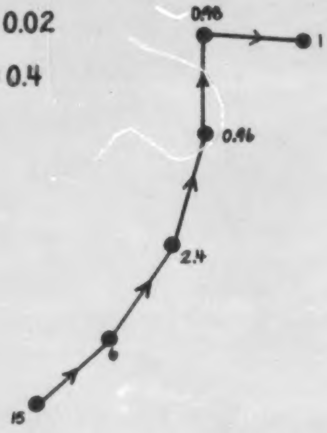
---

Name ______________________ **N 17** ***

Build a red-blue road from 15 to 1.

+0.02

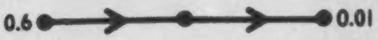
×0.4


0.98
1
0.96
2.4
6
15

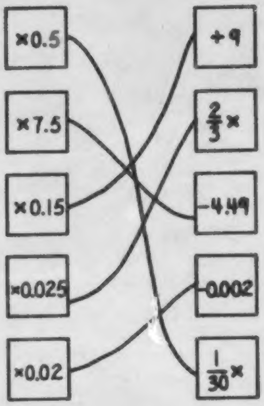
(Other solutions are possible.)

---

Name ______________________ **N 18** ****



Pair the tags.


×0.5
×7.5
×0.15
×0.025
×0.02
÷9
$\frac{2}{3}$×
−4.49
−0.002
$\frac{1}{30}$×

## 2.3 Addition of Rational Numbers

CAPSULE LESSON SUMMARY

Do addition calculations such as "$\frac{1}{5} + \frac{1}{4}$" and "$\frac{3}{4} + \frac{1}{6}$" first by using appropriate partitions of regions. Then do such calculations only thinking about the partitions, but choosing appropriate (useful) fractional names for the numbers being added.

MATERIALS

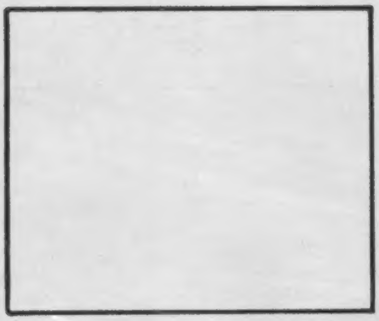
Teacher: Colored chalk
Student: Paper; colored pencils; Worksheets N19, N20, and N21

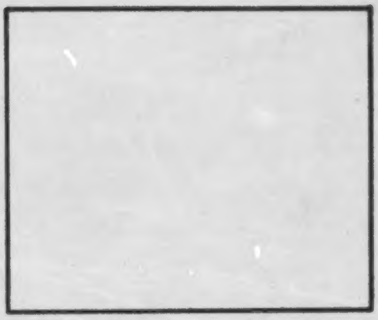
### DESCRIPTION OF LESSON

**Exercise 1**

Write this calculation on the board. Draw two rectangular regions the same size below the calculation.

$$\frac{1}{5} + \frac{1}{4} =$$



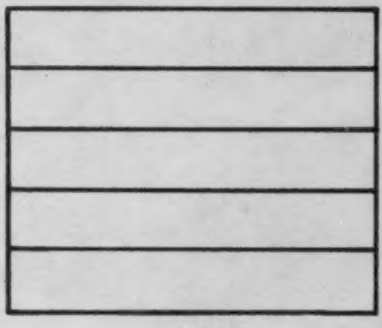


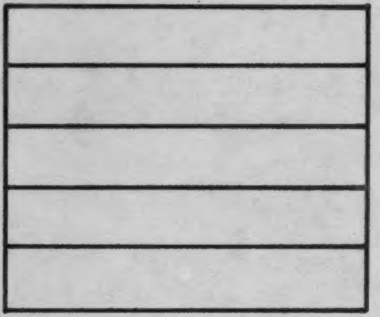
T: Let's do this addition problem with fractions using the method of dividing the regions into rows and columns. How many rows . . . ?

# N 9

S: Five[*] rows, because one of the numbers being added is $\frac{1}{5}$.

Ask a student to make the row partition of each region.

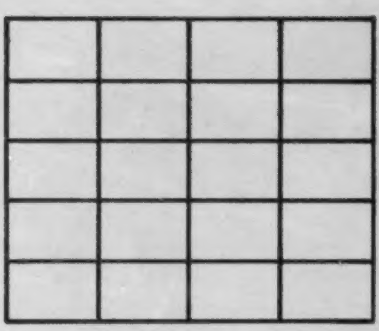


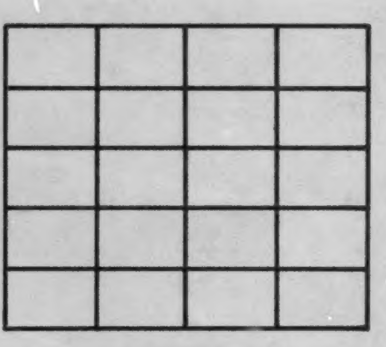


T: How many columns... ?

S: Four columns, because $\frac{1}{4}$ is the other number being added.

Ask a student to make the column partition of each region.





T: How many little pieces are there in each region?

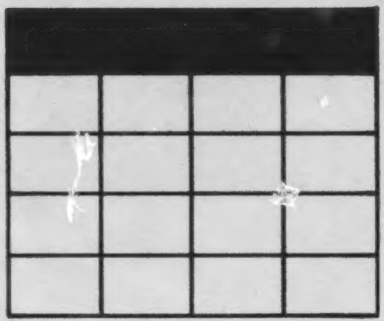
S: 20, since 5 × 4 = 20.

T (pointing to the region at the left): Color one-fifth of this region.

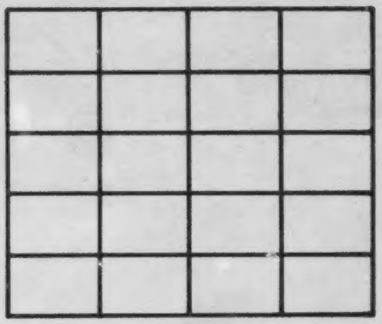
---

[*] Four rows and five columns could also be used.

Invite a student to do the coloring and then to write "$\frac{1}{5}$" below the region.



$\frac{1}{5}$



T: What fractional part of the whole region is each little piece?

S: One-twentieth.

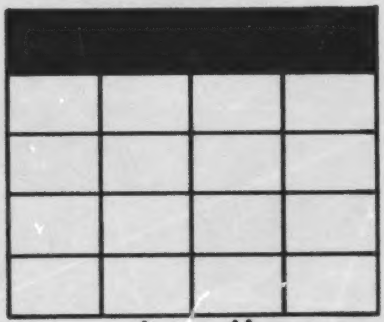
T: How many little pieces are colored?

S: Four.

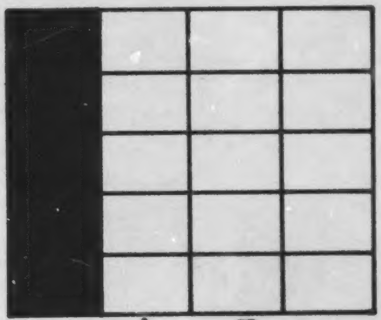
T: Does this picture suggest another name for $\frac{1}{5}$?

S: Yes, "$\frac{4}{20}$".

Add this information to your drawing. In a similar manner, direct the class to color one-fourth of the region at the right and to conclude that $\frac{1}{4} = \frac{5}{20}$.



$\frac{1}{5} = \frac{4}{20}$



$\frac{1}{4} = \frac{5}{20}$

**T:** Which is larger, $\frac{1}{5}$ or $\frac{1}{4}$? Why?

**S:** $\frac{1}{4}$ is larger; five pieces are colored for $\frac{1}{4}$ compared to four pieces for $\frac{1}{5}$.

**T:** How many little pieces are colored in all?

**S:** Nine.

**T:** We said each little piece was one-twentieth of the region, so what fraction do we write for the colored regions together?

**S:** "$\frac{9}{20}$".

Complete the calculation on the board.

$$\frac{1}{5} + \frac{1}{4} = \frac{4}{20} + \frac{5}{20} = \frac{9}{20}$$

Exercise 2

Write this calculation on the board.

$$\frac{3}{4} + \frac{1}{6} =$$

**T:** Our method of doing addition problems with fractions changes the denominators (bottom numbers) to the same number. Let's use this method, but only think about the drawings.

What are some other fractions for $\frac{3}{4}$?

Write names for $\frac{3}{4}$ on the board as students suggest them.

$$\frac{3}{4}=\frac{6}{8}=\frac{12}{16}=\frac{9}{12}=\frac{15}{20}=\frac{24}{32}=\frac{18}{24}=\cdots$$

T: What are some other fractions for $\frac{1}{6}$?

Write these names on the board also.

$$\frac{1}{6}=\frac{2}{12}=\frac{4}{24}=\frac{3}{18}=\frac{6}{36}=\frac{8}{48}=\frac{7}{42}=\cdots$$

T: Which names for $\frac{3}{4}$ and $\frac{1}{6}$ would be useful in doing the addition calculation?

Follow the students' suggestions to do the calculation. Look at more than one way if suggested.

S: We could use "$\frac{9}{12}$" and "$\frac{2}{12}$".

$$\frac{3}{4}+\frac{1}{6}=\frac{9}{12}+\frac{2}{12}=$$

T: What number is $\frac{9}{12}+\frac{2}{12}$?

S: $\frac{11}{12}$.

$$\frac{3}{4}+\frac{1}{6}=\frac{9}{12}+\frac{2}{12}=\frac{11}{12}$$

S: We could use "$\frac{18}{24}$" and "$\frac{4}{24}$".

$$\frac{3}{4}+\frac{1}{6}=\frac{18}{24}+\frac{4}{24}=$$

# N 9

T: What number is $\frac{18}{24} + \frac{4}{24}$?

S: $\frac{22}{24}$

$$\frac{3}{4} + \frac{1}{6} = \frac{18}{24} + \frac{4}{24} = \frac{22}{24}$$

S: $\frac{22}{24} = \frac{11}{12}$

T: We get the same result regardless of which names we use.

Write this calculation on the board.

$$\frac{7}{3} + \frac{5}{8} =$$

Ask the students to copy this problem on their papers and to solve it by finding other names for $\frac{7}{3}$ and $\frac{5}{8}$. Check the problem collectively. The class should conclude that $\frac{7}{3} + \frac{5}{8} = 2\,\frac{23}{24}$.

$$\frac{7}{3} = \frac{14}{6} = \frac{21}{9} = \frac{49}{21} = \frac{56}{24} = \cdots$$

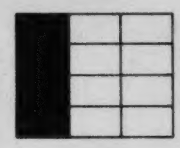
$$\frac{5}{8} = \frac{10}{16} = \frac{20}{32} = \frac{25}{40} = \frac{15}{24} = \cdots$$

$$\frac{7}{3} + \frac{5}{8} = \frac{56}{24} + \frac{15}{24} = \frac{71}{24} = 2\frac{23}{24}$$

Worksheets N19, N20, and N21 are available.

Name ____________________ **N 19 ***

Color one-third of this region red and fill in the box.



$\frac{1}{3} = \frac{\boxed{4}}{12}$

Color three-fourths of this region red and fill in the box.

$\frac{3}{4} = \frac{\boxed{9}}{12}$

Complete the calculation.

$$\frac{1}{3} + \frac{3}{4} = \frac{\boxed{4}}{12} + \frac{\boxed{9}}{12} = \frac{15}{12}$$

(Other colorings are possible. Different names for $\frac{13}{12}$ could be given; e.g., $\frac{13}{12} = 1\frac{1}{12} = \frac{26}{24} = 1\frac{2}{24} = \ldots$.)

---

Name ____________________ **N 20 ****

Write at least five fractions for $\frac{1}{4}$.

$$\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16} = \frac{5}{20} = \frac{6}{24} = \cdots$$

Write at least five fractions for $\frac{8}{5}$.

$$\frac{8}{5} = \frac{16}{10} = \frac{24}{15} = \frac{32}{20} = \frac{40}{25} = \frac{48}{30} = \frac{56}{35} = \cdots$$

Complete the calculation.

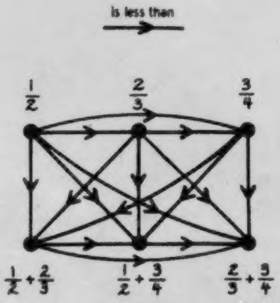
$$\frac{1}{4} + \frac{8}{5} = \frac{5}{20} + \frac{32}{20} = \frac{37}{20}$$

(Other correct names for $\frac{1}{4} + \frac{8}{5}$ could be given; e.g., $\frac{37}{20} = 1\frac{17}{20}$.)

---

Name ____________________ **N 21 *****

Draw as many red arrows as possible between these dots.

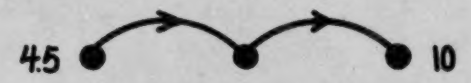


# N 14

## 2.4 Decimals #2

**CAPSULE LESSON SUMMARY**

Solve several sets of problems involving multiplication and division of decimal numbers. Observe some patterns, use the idea of money, and look at related problems with fractions to support the calculations with decimals. Find pairs of functions that the blue and red arrows could be for in situations such as:



**MATERIALS**

Teacher: Colored pencils
Student: Paper; pencil; Worksheets N22, N23, N24, and N25

### DESCRIPTION OF LESSON

Exercise 1

Write these problems on the board.

$10 \times 2.5 =$

$10 \times 0.25 =$

$100 \times 0.25 =$

$10 \times 0.025 =$

$100 \times 0.025 =$

T: Copy these problems and complete the calculations.

Observe the students' work and give help where necessary. Encourage students who have difficulty to think about money. For example, "10 x 2.5" could be thought of as ten times two dollars and fifty cents. Some students may question 0.025, in which case lead a brief discussion of this number.

**T:** How much money is $0.025?

**S:** Between two cents and three cents.

**S:** Two and one-half cents.

Check the problems collectively (answers are provided below) and invite students to observe patterns in the problem set. One observation is suggested below.

$$10 \times 2.5 = 25$$

$$10\times\left(\begin{array}{rcl} 10 \times 0.25 &=& 2.5 \\ 100 \times 0.25 &=& 25 \end{array}\right)10\times$$

$$10\times\left(\begin{array}{rcl} 10 \times 0.025 &=& 0.25 \\ 100 \times 0.025 &=& 2.5 \end{array}\right)10\times$$

Continue in this manner using the sets of problems provided below or others more appropriate for your class. Answers are in the boxes. Use money as a support for the calculations whenever necessary.

$$\begin{array}{rcl} 45 \div 10 &=& \boxed{4.5} \\ 4.5 \div 10 &=& \boxed{0.45} \\ 0.45 \div 10 &=& \boxed{0.045} \\ 4.5 \div 100 &=& \boxed{0.045} \end{array}$$

$$\begin{array}{rcl} 2 \times \boxed{2.5} &=& 5 \\ 20 \times \boxed{0.25} &=& 5 \\ 200 \times \boxed{0.025} &=& 5 \\ 0.2 \times \boxed{25} &=& 5 \\ 0.02 \times \boxed{250} &=& 5 \end{array}$$

$$\begin{array}{rcl} 3 \times 0.8 &=& \boxed{2.4} \\ 30 \times 0.8 &=& \boxed{24} \\ 300 \times 0.8 &=& \boxed{240} \\ 0.3 \times 0.8 &=& \boxed{0.24} \\ 0.03 \times 0.8 &=& \boxed{0.024} \end{array}$$

## N 14

You may prefer to solve the last two problems collectively.

$$0.3 \times 0.8 =$$

$$0.03 \times 0.8 =$$

T: How can we do these calculations?

S: Use fractions.

T: Give me a fraction for 0.3.

S: "$\frac{3}{10}$".

T: ... a fraction for 0.8?

S: "$\frac{8}{10}$".

Write this information on the board.

$$0.3 \times 0.8 = \frac{3}{10} \times \frac{8}{10}$$

T: What number is $\frac{3}{10} \times \frac{8}{10}$?

S: $\frac{24}{100}$.

$$0.3 \times 0.8 = \frac{3}{10} \times \frac{8}{10} = \frac{24}{100}$$

T: Can you give a decimal name for $\frac{24}{100}$?

S: "0.24".

$$0.3 \times 0.8 = \frac{3}{10} \times \frac{8}{10} = \frac{24}{100} = 0.24$$

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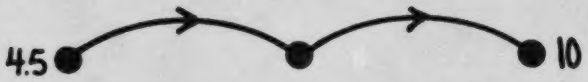
Discuss the second problem in a similar manner. The class should conclude:

$$0.03 \times 0.8 = \frac{3}{100} \times \frac{8}{10} = \frac{24}{1{,}000} = 0.024$$

**Exercise 2**

Draw this arrow picture on the board.



T: If the blue arrow were for + 2.8, what could the red arrow be for?

S: + 2.7, because the middle dot would be for 7.3 and 7.3 + 2.7 = 10.

Continue this activity using the functions suggested in the table below. Possible student responses are also provided.

| → | → |
|---|---|
| −1.7 | +7.2 |
| 3× | −3.5 |
| ×0.3 | +8.65 |
| ÷10 | +9.55 |

Then reverse the situation, asking:

T: If the red arrow were for + 6.3, what could the blue arrow be for?

If a suggestion for the blue arrow is not forthcoming, ask:

**T:** Which number is here (point to the middle dot)? Why?

**S:** 3.7, because 10 - 6.3 = 3.7.

Label the dot "3.7".

**T:** Now what could the blue arrow be for?

**S:** -0.8.

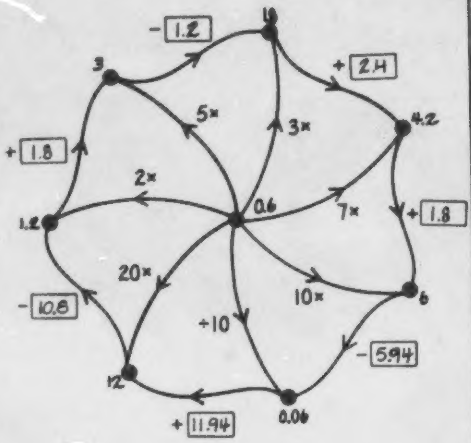
Continue this activity using the functions suggested in the table below. Some student responses are also provided.

| → | → |
|---|---|
| −2 or $\frac{5}{4}x$ | $4x$ |
| −4.25 or ÷18 | $40x$ |
| +20.5 or $\frac{82}{4}x$ | $0.4x$ |

Worksheets N22, N23, N24, and N25 are available.

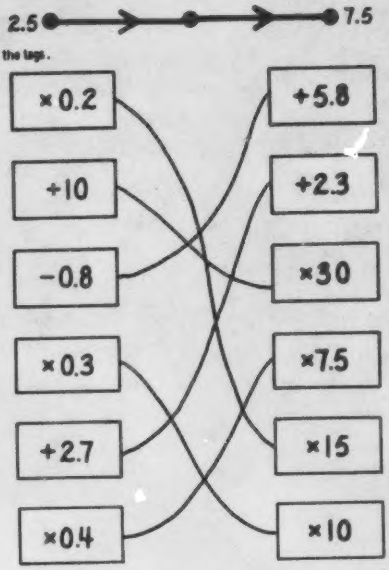
Name ____________________ **N 22 ***

Label the dots and fill in the boxes for the arrows.


4
3
4.2
1.2
0.6
6
12
0.06
− 1.2
+ 2.4
+ 1.8
+ 1.8
− 10.8
− 5.94
+ 11.94
5×
3×
2×
7×
20×
10×
÷10

---

Name ____________________ **N 23 ****


2.5
7.5
Pair the tags.
×0.2
÷10
−0.8
×0.3
+2.7
×0.4
+5.8
+2.3
×30
×7.5
×15
×10

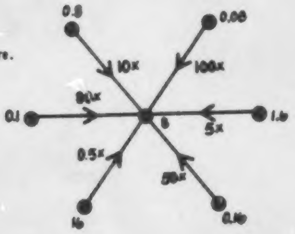
---

Name ____________________ **N 24 *****

Rack is a secret number.

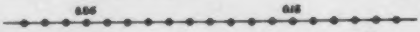
First clue

Rack is in this picture.


0.8
0.08
0.1
1.6
16
0.16
8
10×
100×
80×
5×
0.5×
50×

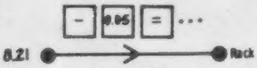
Second clue

One of these dots is for Rack.



Rack could be 0.08, 0.16, or 0.1.

Third clue


−
0.05
=
...
0.21
Rack

Rack is 0.16.

---

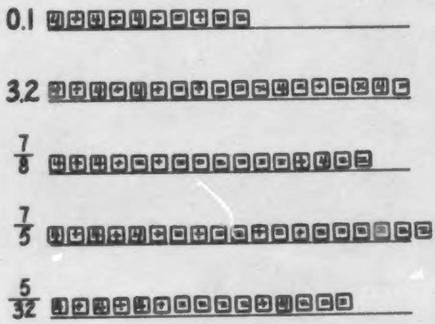
Name ____________________ **N 25 ******

Put each number on the hand-calculator using just these keys:

4 + − × ÷ =

Write the keys in the order you use them. You may use a key more than once.

It costs 1¢ each time you press a key. Try not to spend more than 20¢ for each number.


0.1
3.2
7/8
7/5
5/32

(Many solutions are possible. Refer to Section 2(f) of the General Introduction for a discussion of hand-calculator characteristics which could affect the solutions.)

# N 18

## 2.5 Decimals #3

**CAPSULE LESSON SUMMARY**

Play a Guess My Rule game with a secret operation * where a * b = (a x b) + 1. Locate decimals on the number line when non-standard names such as "1/2 x 0.5", "0.5 x 0.3", and "2 - 0.95" are given for the numbers. In the same way, place decimals in a string picture with one string labeled "larger than 0.2".

**MATERIALS**

Teacher: Colored chalk
Student: Paper; pencil; Worksheets N26, N27, and N28

DESCRIPTION OF LESSON

Write this information on the board.

$$3 * 4 = 13$$
$$10 * 3 = 31$$
$$2 * 0.8 = \boxed{\phantom{00}}$$

T: I have a secret rule for *. Here are two clues about my rule. Can you guess my rule? Which number do you think goes in the box?

NOTE: The rule is: a * b = (a x b) + 1. The number in the box is 2.6, because 2 * 0.8 = (2 x 0.8) + 1 = 1.6 + 1 = 2.6. Do not tell the class the rule at this time.

Suggest students whisper their guesses in your ear. Acknowledge, aloud, any correct guess and deny any incorrect one. For example, "Yes, that's correct" or "No, 1.7 is not the number." Let a student who guesses correctly tell the class that 2.6 is in the box but do not give away the rule. If no one guesses correctly, put 2.6 in the box yourself.

Continue the activity with these problems (answers are in the boxes).

$$5 * 6 = \boxed{31}$$
$$5 * 0.6 = \boxed{4}$$
$$0.5 * 0.6 = \boxed{1.3}$$
$$6 * 7 = \boxed{43}$$
$$8 * 5 = \boxed{41}$$
$$0.8 * 5 = \boxed{5}$$
$$0.8 * 0.5 = \boxed{1.4}$$

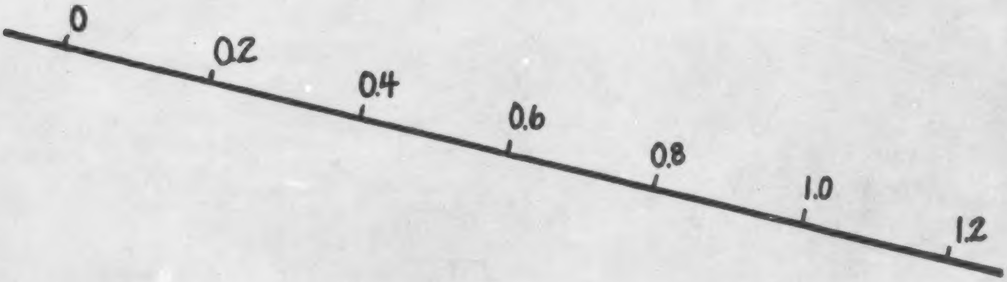
T: Who can explain the rule?

S: Multiply the two numbers, then add 1 to the result. For example, $5 * 6 = (5 \times 6) + 1 = 30 + 1 = 31$.

Distribute worksheet N26 and instruct the students to fill in the boxes. There are problems on both sides of the worksheet; however, when most students have completed the first side, check it collectively. See the answer key on page 72.

Exercise 2

Draw this part of the number line on the board and ask the students to copy it on their papers.

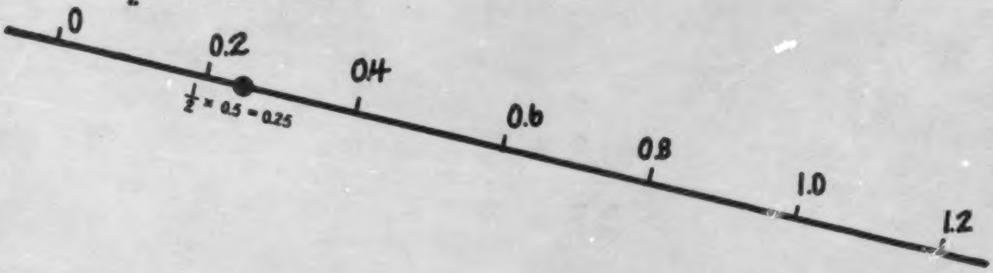


# N 18

**T:** I shall write a number on the board. Locate this number on your number line.

Write "$\frac{1}{2} \times 0.5$" on the board. Give students time to locate this number; then invite a student to locate $\frac{1}{2} \times 0.5$ at the board.

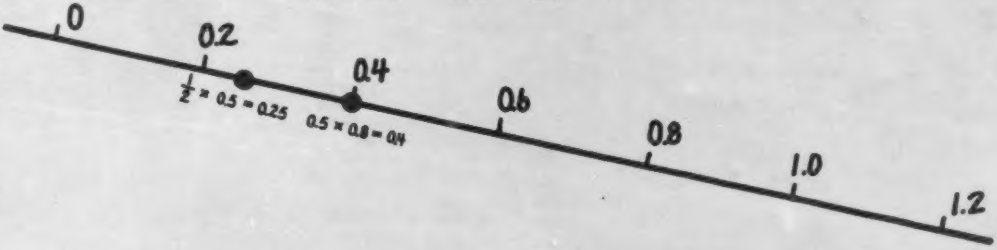
**S:** $\frac{1}{2} \times 0.5 = 0.25$.


0
0.2
0.4
0.6
0.8
1.0
1.2
$\frac{1}{2} \times 0.5 = 0.25$

Write "0.5 x 0.8" on the board and instruct the students to locate this number.

**S:** 0.5 x 0.8 = 0.4, because one-half of eighty cents is forty cents.

**S:** $0.5 \times 0.8 = \frac{5}{10} \times \frac{8}{10} = \frac{40}{100} = \frac{4}{10} = 0.4$.


0
0.2
0.4
0.6
0.8
1.0
1.2
$\frac{1}{2} \times 0.5 = 0.25$
$0.5 \times 0.8 = 0.4$

Continue this activity instructing students to locate:

$$0.5 \times 0.3$$

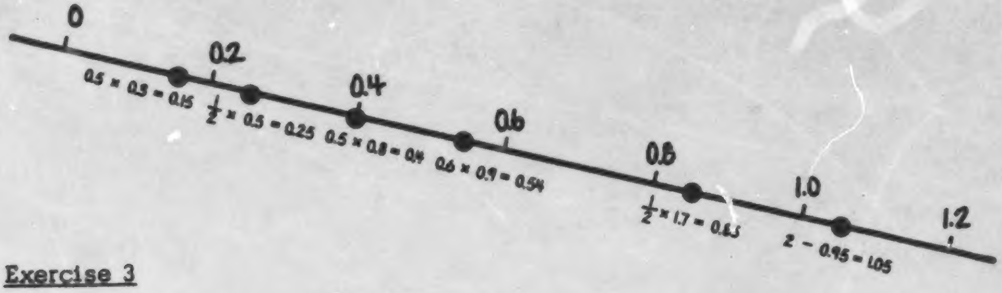
$$0.6 \times 0.9$$

$$\frac{1}{2} \times 1.7$$

$$2 - 0.95$$

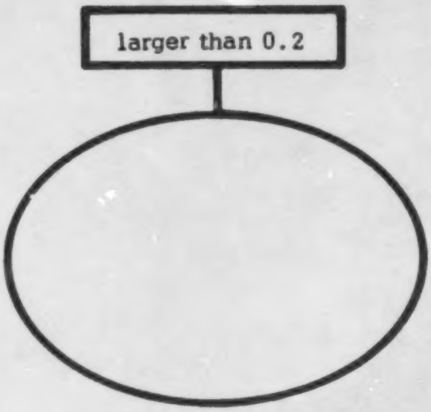
Ask students to locate each number at the board. Be prepared to discuss the calculations involved. Many difficulties that students might have can be solved by using the idea of money as a support.

Your completed number line should look something like this:


0
0.2
0.4
0.6
0.8
1.0
1.2
0.5 × 0.3 = 0.15
½ × 0.5 = 0.25
0.5 × 0.8 = 0.4
0.6 × 0.9 = 0.54
½ × 1.7 = 0.85
2 − 0.95 = 1.05

Exercise 3

Draw this string picture on the board.


larger than 0.2

T: Copy this string picture on your paper. I shall write some numbers on the board. Put these numbers in the string picture.

Write these numbers on the board.

0.5 × 0.2
3 × 0.07
10 × 0.01

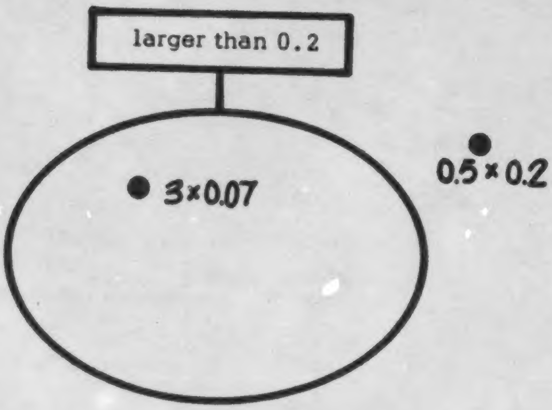
## N 18

Allow a few minutes for individual work before leading a collective discussion.

S: 0.5 x 0.2 = 0.1 and 0.1 is smaller than 0.2.

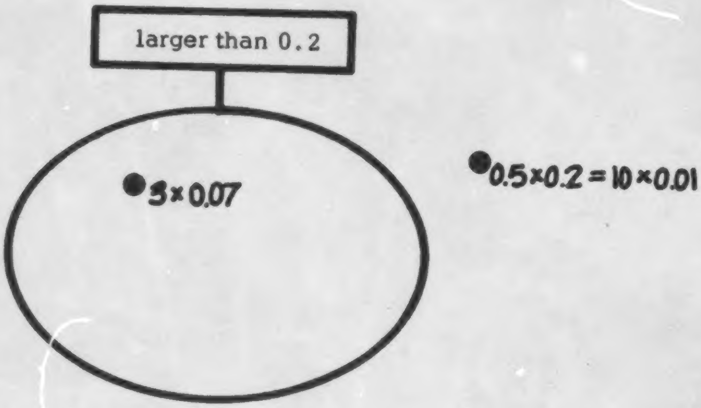
S: 3 x 0.07 = 0.21 and 0.21 is larger than 0.2.

Invite the students to put the numbers in the string picture.


larger than 0.2
3×0.07
0.5×0.2

S: 10 x 0.01 = 0.1 but there is already a dot for 0.1.

Label the dot in this manner.


larger than 0.2
3×0.07
0.5×0.2 = 10×0.01

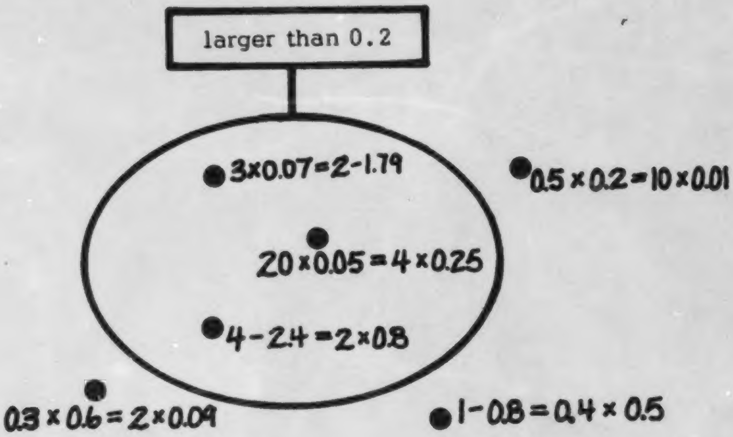
Continue this activity by asking the students to put these numbers in the string picture.

$1 - 0.8$ $\quad$ $0.3 \times 0.6$ $\quad$ $20 \times 0.05$

$4 - 2.4$ $\quad$ $0.4 \times 0.5$ $\quad$ $2 \times 0.09$

$2 - 1.79$ $\quad$ $2 \times 0.8$ $\quad$ $4 \times 0.25$

It may be necessary to remind some students that a dot may be labeled in more than one way. Check the work collectively by asking students to complete the picture at the board.


larger than 0.2
3×0.07=2−1.79
0.5×0.2=10×0.01
20×0.05=4×0.25
4−2.4=2×0.8
0.3×0.6=2×0.09
1−0.8=0.4×0.5

Worksheets N27 and N28 are available.

Name ______________________ **N 26**

Fill in the boxes.

$\boxed{a * b = (a \times b) + 1}$

$5 * 8 = \boxed{41}$

$7 * 7 = \boxed{50}$

$3 * 8 = \boxed{25}$

$5 * 11 = \boxed{56}$

$20 * 3 = \boxed{61}$

$25 * 4 = \boxed{101}$

$6 * \boxed{8} = 49$

$\boxed{6} * 9 = 55$

$8 * \boxed{4} = 33$

$\boxed{11} * 4 = 45$

---

Fill in the boxes.

$\boxed{a * b = (a \times b) + 1}$

$2 * 0.7 = \boxed{2.4}$

$5 * 0.6 = \boxed{4}$

$6 * 7 = \boxed{43}$

$6 * 0.7 = \boxed{5.2}$

$8 * 5 = \boxed{41}$

$0.8 * 5 = \boxed{5}$

$0.8 * 0.5 = \boxed{1.4}$

$10 * \boxed{0.6} = 7$

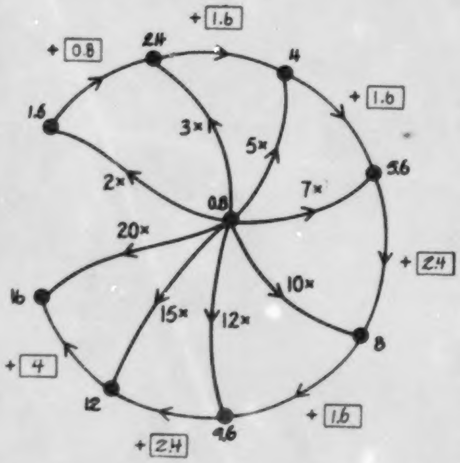
$\boxed{0.5} * 4 = 3$

$8 * \boxed{0.3} = 3.4$

---

Name ______________________ **N 27** *

Label the dots and fill in the boxes for the arrows.


+ 1.6
+ 0.8
2.4
4
1.6
+ 1.6
3×
5×
5.6
2×
0.8
7×
20×
+ 2.4
10×
16
15×
12×
8
+ 4
12
+ 1.6
9.6
+ 2.4

---

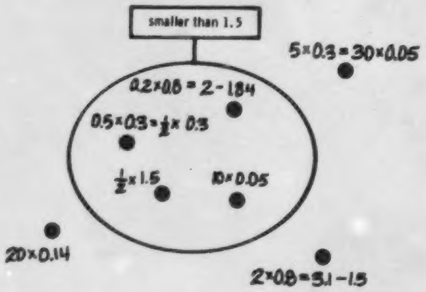
Name ______________________ **N 28** **

Put these numbers in the string picture.

$5 \times 0.3$ $0.5 \times 0.3$ $2 \times 0.8$ $0.2 \times 0.8$

$\frac{1}{2} \times 0.3$ $30 \times 0.05$ $\frac{1}{2} \times 1.5$

$3.1 - 1.5$ $2 - 1.84$ $10 \times 0.05$ $20 \times 0.14$


smaller than 1.5
5×0.3 = 30×0.05
0.2×0.8 = 2 − 1.84
0.5×0.3 = ½× 0.3
½×1.5
10×0.05
20×0.14
2×0.8 = 3.1 − 1.5

## 2.6 Decimals #4

CAPSULE LESSON SUMMARY

Label the dots in an arrow picture where the arrows are all for multiplication by decimals. Use the picture and appropriate return arrows to do some division calculations with decimals. Label dots in an arrow picture where the arrows are for +, -, x, or ÷ decimals. Introduce the idea of building "one-digit" arrow roads, that is, roads where the arrows must be for +, -, x, or ÷ some one-digit positive whole number and build some such arrow roads.

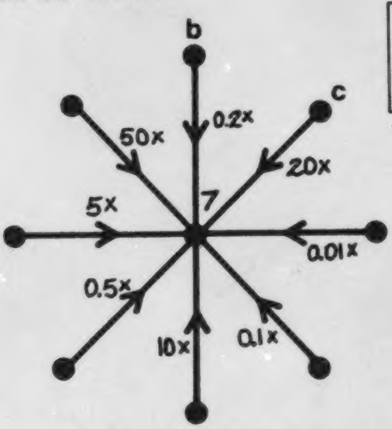
MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

### DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Copy this picture and label the dots.

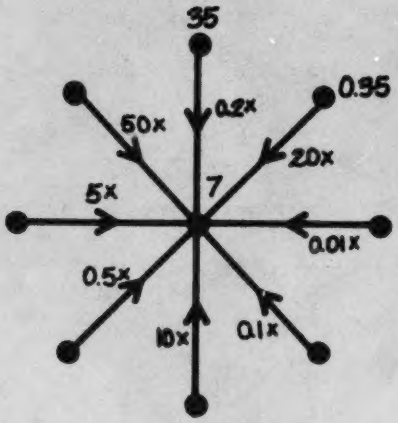
Let the students work independently for a while. When most students have finished, call on volunteers to label the dots at the board.

T (pointing to b): Which number is here?

S: 35, because 0.2 x 35 = 7.

T (pointing to c): Which number is here?

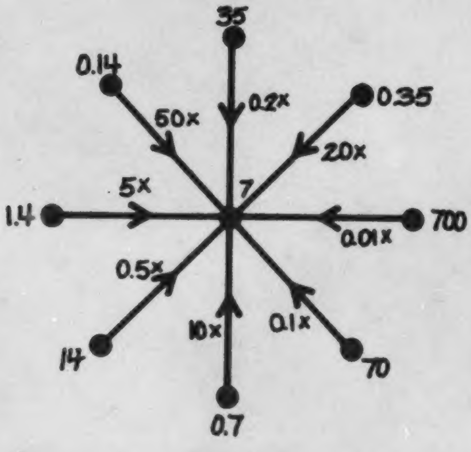
S: 0.35, because 20 x 0.35 = 7.


35
0.35
0.2x
50x
20x
7
5x
0.01x
0.5x
10x
0.1x

T: Do you notice any interesting pattern?

S: 20 is 100 times 0.2 and 35 is 100 times 0.35.

Continue to let students label the dots and to make any comments about patterns they observe in the arrow picture. A completed picture follows.


35
0.14
0.2x
0.35
50x
20x
5x
7
1.4
700
0.01x
0.5x
10x
0.1x
14
70
0.7

Write this calculation on the board close to the arrow picture.

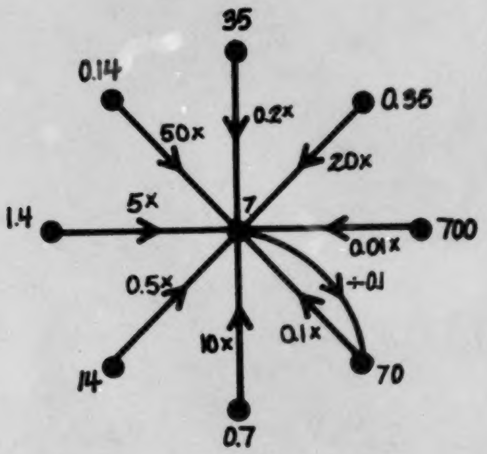
$$7 \div 0.1 =$$

T: Copy this problem and solve it. You may use the arrow picture to help you.

Allow a few minutes for individual work before discussing the problem collectively.

S: Use the return arrow for 0.1 x .

Call on a student to draw the return arrow starting at 7 and ending at 70 and to label it "÷0.1".



S: $7 \div 0.1 = 70$.

Write these problems on the board and ask the students to solve them. Check the problems collectively by looking at the appropriate return arrows. Answers are in the boxes.

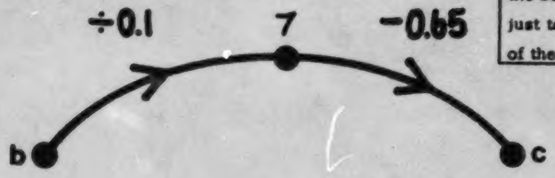
$$7 \div 0.2 = \boxed{35} \qquad 7 \div 0.01 = \boxed{700}$$

$$7 \div 20 = \boxed{0.35}$$

Exercise 2

Draw this arrow picture on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T (pointing to c) : Which number is here?

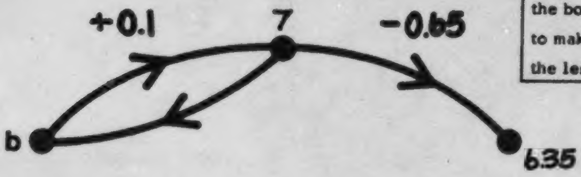
S : 6.35, 7 - 0.65 = 6.35.

Label the dot "6.35".

T (pointing to b) : Which number is here?

For the benefit of students who have difficulty, draw the return arrow for ÷0.1.

Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

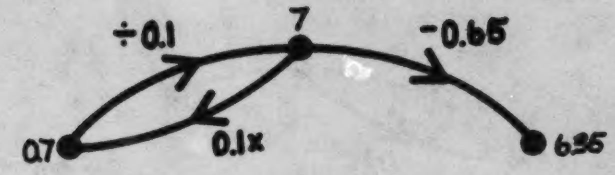


T (tracing the blue return arrow) : What could this return arrow be for?

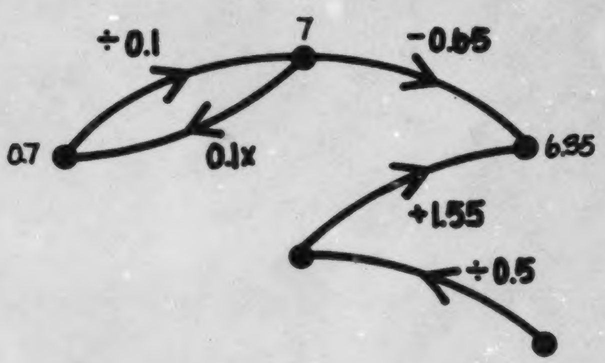
S : x0.1.

T (pointing to b) : Which number is here?

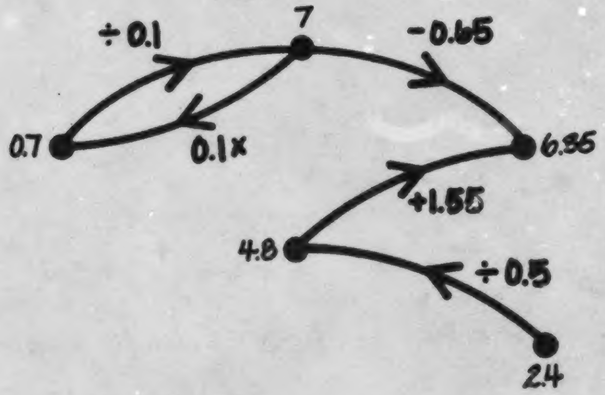
S: 0.7, because 7 x 0.1 = 0.7.


÷0.1
7
-0.65
0.7
0.1x
6.35

Add the two arrows shown in the next illustration to your picture. Instruct the students to copy the picture and to label the dots.

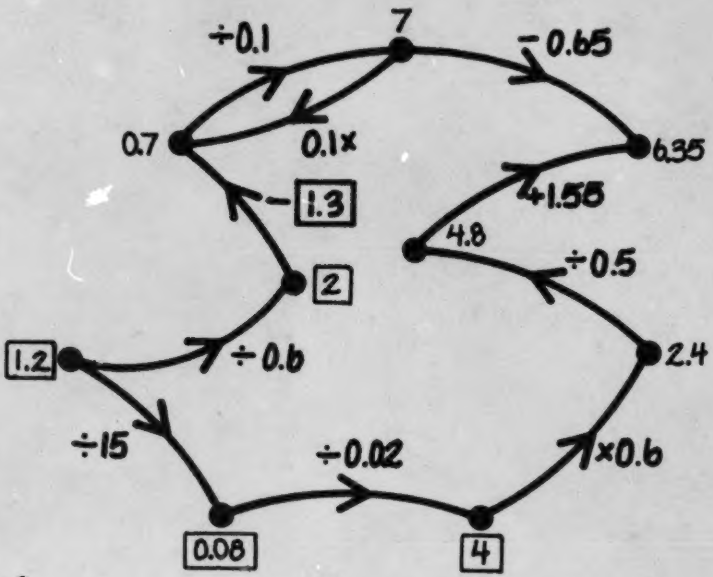

÷0.1
7
-0.65
0.7
0.1x
6.35
+1.55
÷0.5

Encourage students who have difficulty to draw and label return arrows. Check this work by asking students to label the dots at the board.


÷0.1
7
-0.65
0.7
0.1x
6.35
+1.55
4.8
÷0.5
2.4

# N 21

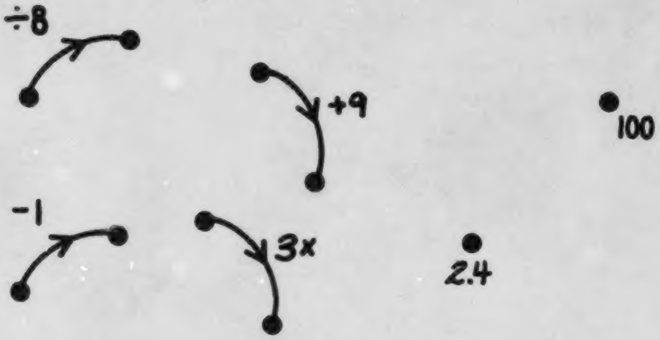
Continue this activity by drawing the arrows shown in the next illustration. Instruct the students to label the dots (answers are in the boxes), and then check the work by asking students to label the dots at the board.


÷0.1
7
-0.65
0.7
0.1×
6.35
-1.3
+1.55
4.8
÷0.5
2
1.2
÷0.6
2.4
÷15
÷0.02
×0.6
0.08
4

Exercise 3

NOTE : Hand-calculators may be used for this exercise.

Write this information on the board.

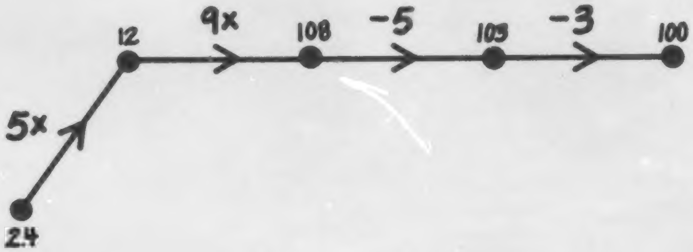


T : Red arrows can be for add, subtract, multiply, or divide by some positive one-digit whole number. A few examples of what red arrows could be for are on the board.

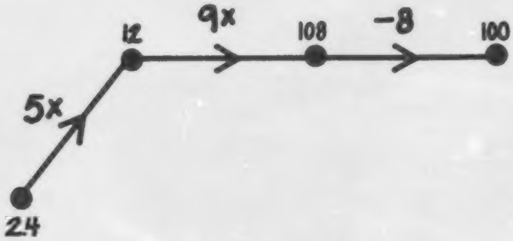
T: Build an arrow road from 2.4 to 100 using these kinds of arrows.

Let the students work independently on this problem for a few minutes. Faster students should be encouraged to find a shortest road. Do not press the idea of a shortest road on the weaker students. Too often in their search for a shortest road weaker students find no road.

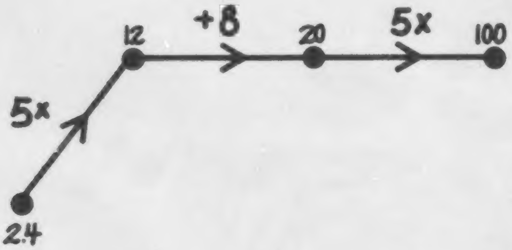
Frequently an arrow road can be shortened by a composition of functions. For example,



can be shortened to . . .

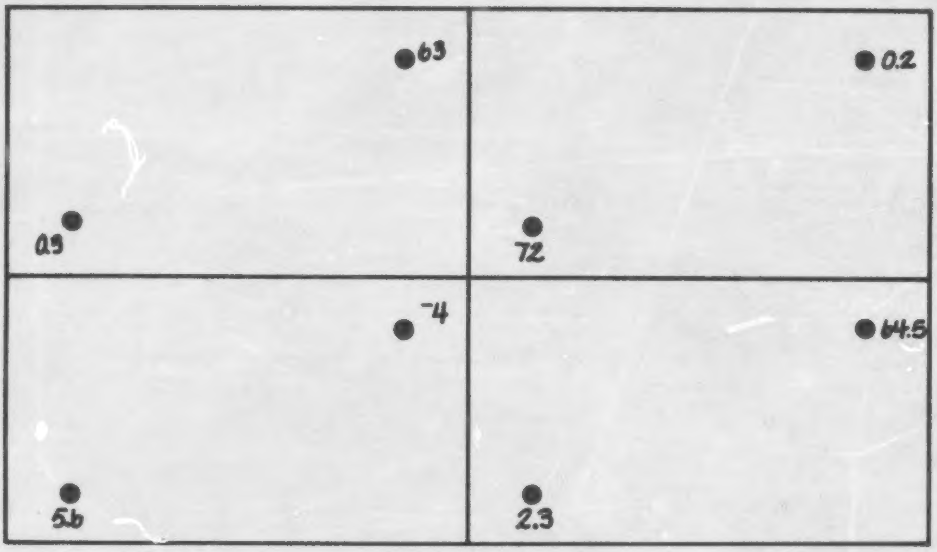


When most students have at least one solution, call on volunteers to draw their roads on the board. There are many possible solutions; another shortest road is given below.

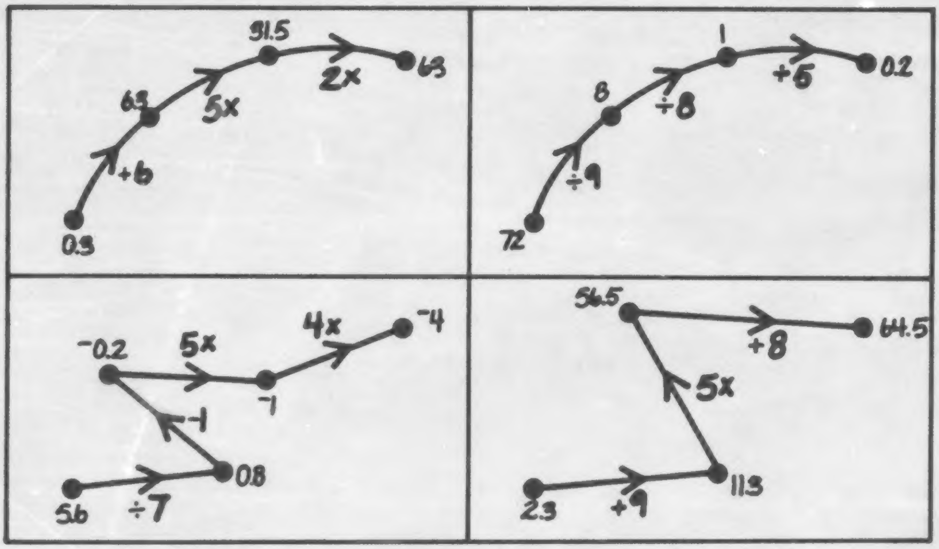


# N 21

Continue this activity using the starting and ending numbers suggested below or others more appropriate for your class.


63
0.3
0.2
72
-4
5.6
64.5
2.3

Check the problems collectively by asking students to put solutions on the board. Many solutions are possible. A shortest road for each problem is given below.


31.5
6.3
63
+6
5x
2x
0.3
1
8
0.2
÷9
÷8
÷5
72
-0.2
5x
4x
-4
-1
-1
0.8
5.6
÷7
56.5
64.5
+8
5x
11.3
2.3
+9

## 2.7 Guess My Rule

CAPSULE LESSON SUMMARY

Find many fractional names for numbers such as $\frac{3}{5}$ and $\frac{5}{2}$.

Play a Guess My Rule game where the rule involves a square configuration of numbers.

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \qquad a \times d = b \times c \text{ or } \frac{a}{c} = \frac{b}{d} \text{ or } \frac{a}{b} = \frac{c}{d}$$

Use an arrow picture to look at the relationship between the numbers in such a configuration.

MATERIALS

Teacher: Colored chalk

Student: Paper; colored pencils; Worksheets N29, N30, and N31

DESCRIPTION OF LESSON

### Exercise 1

Write "$\frac{3}{5}$" on the board and ask:

T: Can you suggest some other fractions for $\frac{3}{5}$?

As students suggest names for $\frac{3}{5}$ write them on the board in a string of equalities.

$$\frac{3}{5} = \frac{6}{10} = \frac{12}{20} = \frac{21}{35} = \cdots$$

After several names for $\frac{3}{5}$ have been suggested, ask the class to fill in the boxes for other names; for example,

$$\frac{3}{5}=\frac{6}{10}=\frac{12}{20}=\frac{21}{35}=\frac{9}{\square}=\frac{\square}{50}=\frac{24}{\square}=\frac{\square}{25}$$

S: $\frac{3}{5} = \frac{9}{15}$.

S: $\frac{3}{5} = \frac{30}{50}$.

S: $\frac{3}{5} = \frac{24}{40}$.

S: $\frac{3}{5} = \frac{15}{25}$.

Repeat the above activity with $\frac{3}{4}$ and $\frac{5}{2}$. If you prefer, instruct the students to write the names they find on their papers. List some of the suggested names on the board, and ask the class to fill in the boxes for several others.

$$\frac{3}{4}=\frac{6}{8}=\frac{9}{\boxed{12}}=\frac{\boxed{18}}{24}=\frac{15}{\boxed{20}}=\frac{\boxed{24}}{32}$$

$$\frac{5}{2}=\frac{15}{6}=\frac{\boxed{10}}{4}=\frac{25}{\boxed{10}}=\frac{\boxed{20}}{8}=\frac{45}{\boxed{18}}$$

Exercise 2

Write this information on the board.

| | |
|---|---|
| 3 | 4 |
| 6 | 8 |

| | |
|---|---|
| 5 | 10 |
| 15 | |

T: I am thinking of a secret rule for putting numbers in four parts of a square. I will give you some hints and you should try to guess my rule.

NOTE: There are several ways to formulate this rule. For example,

| | |
|---|---|
| a | b |
| c | d |

$$a \times d = c \times b$$

$$\frac{a}{c} = \frac{b}{d}$$

$$\frac{a}{b} = \frac{c}{d}$$

Therefore,

| | |
|---|---|
| 5 | 10 |
| 15 | 30 |

because $5 \times 30 = 15 \times 10 = 150$ or $\frac{5}{15} = \frac{10}{30}$ or $\frac{5}{10} = \frac{15}{30}$. Do not tell the class the rule at this time.

T: Which number do you think belongs in the empty part?

Walk around the room and let students whisper their guesses in your ear. Acknowledge, aloud, any correct guess and deny any incorrect one. For example, "Yes, that's correct" or "No, 20 does not belong in the box." After a while, invite a student to announce that 30 goes in the empty region but not to give away the rule. If necessary, give this information yourself. Continue the activity using the configurations suggested below. In each case ask the students to provide the number in red.

| | |
|---|---|
| 3 | 5 |
| 12 | 20 |

| | |
|---|---|
| 5 | 7 |
| 15 | 21 |

| | |
|---|---|
| 4 | 20 |
| 6 | 30 |

| | |
|---|---|
| 4 | 6 |
| 6 | 9 |

| | |
|---|---|
| 6 | 9 |
| 8 | 12 |

| | |
|---|---|
| 12 | 16 |
| 9 | 12 |

| | |
|---|---|
| 8 | 10 |
| 12 | 15 |

T: Who can explain my rule? There are several ways to explain it.

S: You multiply diagonal numbers and get the same result each time.

S: In the first square, $\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 6 & 8 \\ \hline\end{array}$, $3 \times 8 = 6 \times 4 = 24$. It works for each square on the board.

S: I looked at the columns and thought about fractions. For the second example, $\begin{array}{|c|c|}\hline 5 & 10 \\ \hline 15 & 30 \\ \hline\end{array}$, $\frac{5}{15} = \frac{1}{3} = \frac{10}{30}$.

S: Or, if you look at the rows, $\frac{5}{10} = \frac{1}{2} = \frac{15}{30}$.

Be sure the class understands the rule by asking them to complete these configurations. Answers are in red.

| | |
|---|---|
| 20 | 12 |
| 15 | 9 |

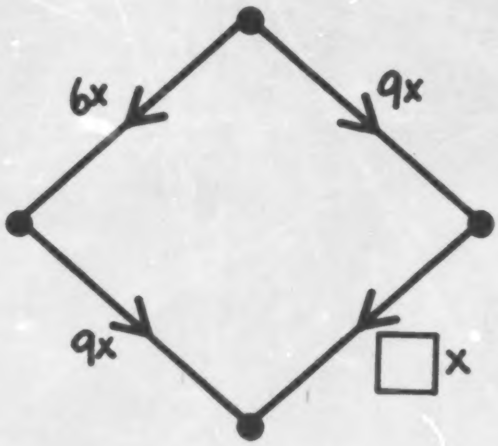
| | |
|---|---|
| 2 | 14 |
| 5 | 35 |

| | |
|---|---|
| 6 | 18 |
| 4 | 12 |

Exercise 3

Put this square configuration of numbers and draw this arrow picture on the board.

| | |
|---|---|
| 6 | 9 |
| 8 | 12 |

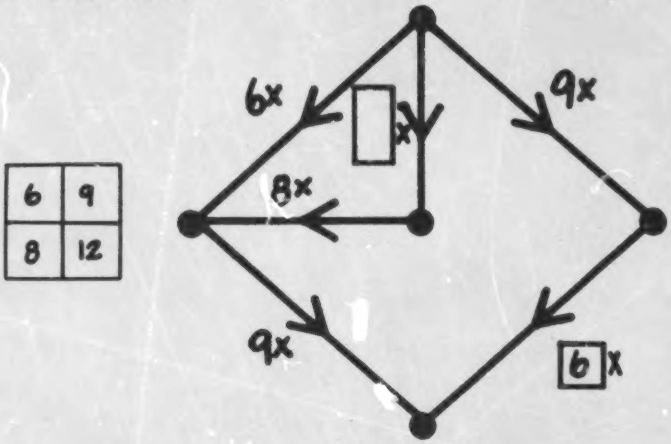


T: Let's see if we can use an arrow picture to help us think about this relationship in another way. Copy this picture on your paper and label the arrow.

After a minute or two, invite a student to label the arrow at the board.

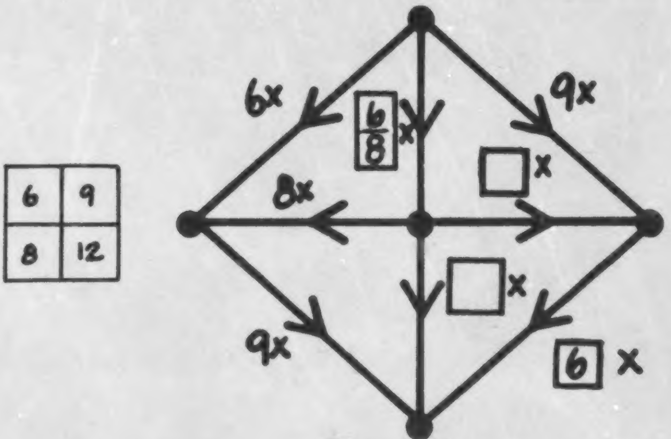
S (tracing the appropriate arrows): 6 x followed by 9 x is the same as 9 x followed by 6 x.

Add these arrows to your picture. Instruct the students to copy them and then to label the new blue arrow.



S (tracing the appropriate arrows): 6 x followed by "against" 8 x is $\frac{6}{8}$ x.

Some students may comment that $\frac{6}{8}$ x = $\frac{3}{4}$ x, however, for this activity be sure to label the blue arrow "$\frac{6}{8}$ x". Add these two arrows to your picture. Instruct the students to copy the new arrows and to label them.



# N 25

S (tracing the appropriate arrows): This blue arrow is for 72x, because 8x followed by 9x is 72x.

S (tracing the appropriate arrows): The red arrow is for 12x, because 72x is 12x followed by 6x.

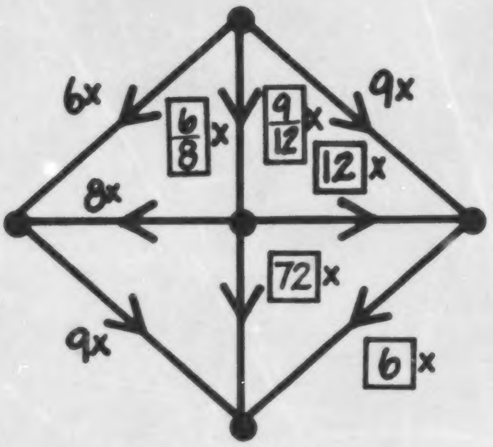
Label the arrows appropriately and direct the class' attention to the upper blue arrow.

T: Can you see another way to label this blue arrow that is suggested by the arrow picture?

S: $\frac{9}{12}$x, because 9x followed by "against" 12x is $\frac{9}{12}$x.

Observe that the labels show the relationship between the numbers in the square configuration. Your picture should look like this upon completion of this activity.

| | |
|---|---|
| 6 | 9 |
| 8 | 12 |



$$\frac{6}{8} = \frac{9}{12}$$

Worksheets N29, N30, and N31 are available.

Name ______________________ **N 29** *

$\frac{a}{b} = \frac{c}{d}$

| | |
|---|---|
| a | c |
| b | d |

a x d = b x c

Complete.

| | |
|---|---|
| 3 | 6 |
| 5 | 10 |

| | |
|---|---|
| 2 | 5 |
| 4 | 10 |

| | |
|---|---|
| 3 | 4 |
| 6 | 8 |

| | |
|---|---|
| 5 | 10 |
| 6 | 12 |

| | |
|---|---|
| 6 | 4 |
| 9 | 6 |

| | |
|---|---|
| 5 | 7 |
| 10 | 14 |

| | |
|---|---|
| 6 | 9 |
| 10 | 15 |

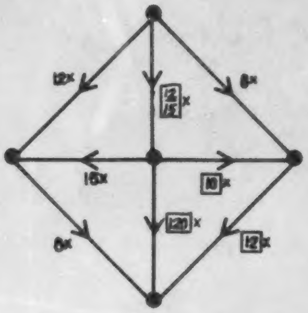
| | |
|---|---|
| 6 | 8 |
| 15 | 20 |

| | |
|---|---|
| 10 | 15 |
| 8 | 12 |

---

Name ______________________ **N 30** **

Fill in the boxes for the arrows.



The picture suggests two ways to fill in the box for the blue arrow.

They are: [illegible] × or [illegible] ×

There are many other ways. Fill in the boxes to show some of them.

[illegible] ×, [illegible] ×, [illegible] ×, [illegible] ×, [illegible] ×, [illegible] ×.

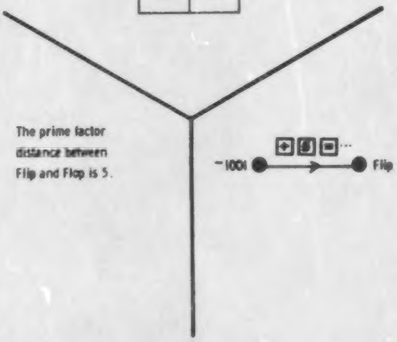
(Other solutions are possible.)

---

Name ______________________ **N 31** ***

Flip and Flop are secret whole numbers.

| | |
|---|---|
| 9 | Flop |
| Flip | 12 |



The prime factor distance between Flip and Flop is 5.

Flip is 4 ; Flop is 27 .

# N 29

## 2.8 String Pictures with Rational Numbers

CAPSULE LESSON SUMMARY

Locate rational numbers in a string picture with one string labeled "larger than $\frac{1}{2}$" when the numbers are given with non-standard names such as "$\frac{2}{3} \times \frac{3}{2}$", "$\frac{3}{8} + \frac{1}{4}$", and "$2 - \frac{5}{4}$". Play a game between two teams where the object is to label the dots in a two-string picture using a set of non-standard names for the numbers.

MATERIALS

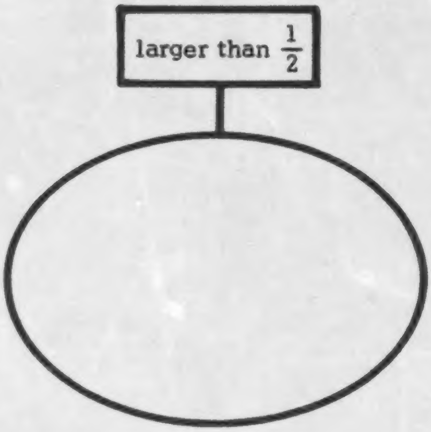
Teacher: Colored chalk

Student: Paper; colored pencils; Worksheets N32, N33, and N34

DESCRIPTION OF LESSON

Exercise 1

Draw this string picture on the board.



T: Copy this string picture on your paper. I'll write some numbers on the board. Put these numbers in your picture.

Write these numbers on the board.

$$\frac{2}{5}$$

$$\frac{1}{3}+\frac{1}{2}$$

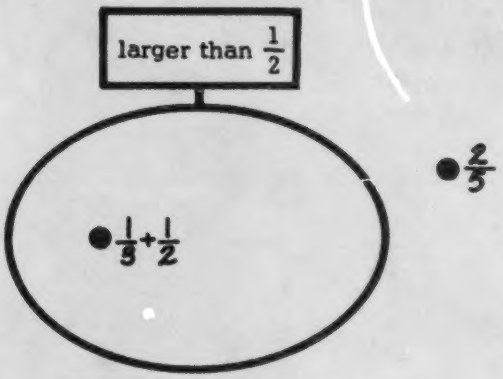
$$0.4$$

Allow a few minutes for individual work before leading a collective discussion. Call on students to locate the numbers in the picture.

S: $\frac{2}{5} = \frac{4}{10}$ and $\frac{1}{2} = \frac{5}{10}$; so $\frac{2}{5}$ is smaller than $\frac{1}{2}$.

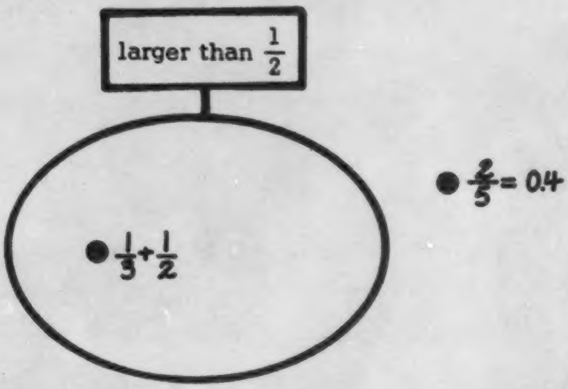
S: $\frac{1}{3} + \frac{1}{2}$ is larger than $\frac{1}{2}$, because $\frac{1}{3}$ is added to $\frac{1}{2}$.

S: $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ and $\frac{5}{6}$ is larger than $\frac{1}{2}$.


larger than $\frac{1}{2}$
$\frac{1}{3}+\frac{1}{2}$
$\frac{2}{5}$

S: $0.4 = \frac{4}{10} = \frac{2}{5}$ and $\frac{2}{5}$ is already in the picture.

Label the dot in this manner.


larger than $\frac{1}{2}$
$\frac{1}{3}+\frac{1}{2}$
$\frac{2}{5}=0.4$

# N 29

Continue this activity by instructing the students to put these numbers, or others more appropriate to the numerical level of your class, in the string picture.

$0.6$

$0.75$

$0.1$

$0.5$

$\frac{1}{2} \times \frac{1}{2}$

$\frac{2}{3} \times \frac{3}{2}$

$\frac{4}{5} \times \frac{3}{4}$

$\frac{1}{2} + \frac{1}{2}$

$\frac{1}{4} + \frac{1}{4}$

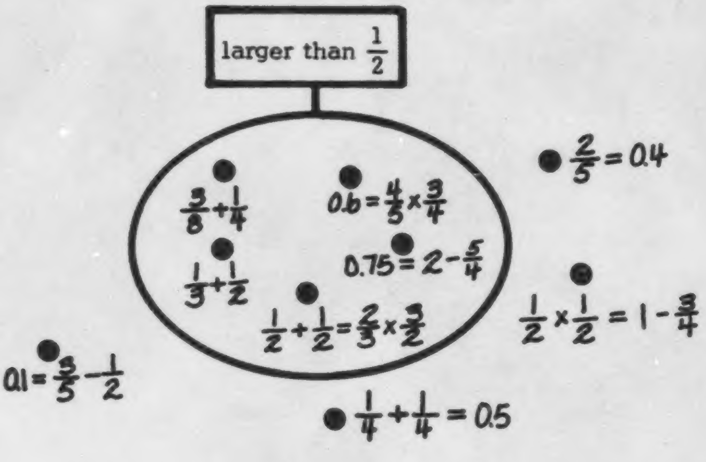
$\frac{3}{8} + \frac{1}{4}$

$1 - \frac{3}{4}$

$2 - \frac{5}{4}$

$\frac{3}{5} - \frac{1}{2}$

It may be necessary to remind some students that a dot may be labeled in more than one way. Check the work collectively by calling on volunteers to locate the numbers in the picture at the board. Be prepared to discuss this situation with the class; this activity offers an opportunity for students to become better acquainted with rational numbers. A completed string picture follows.



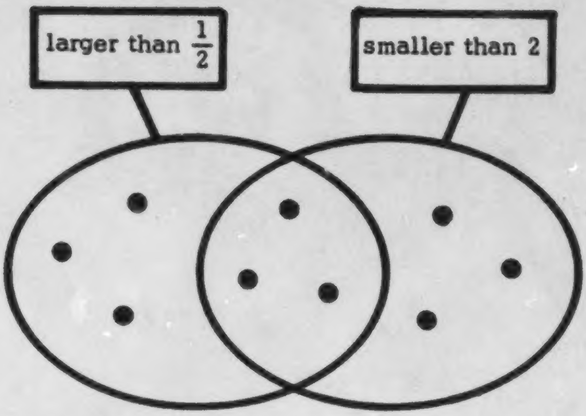
larger than $\frac{1}{2}$
$\frac{3}{8}+\frac{1}{4}$
$0.6=\frac{4}{5}\times\frac{3}{4}$
$\frac{2}{5}=0.4$
$\frac{1}{3}+\frac{1}{2}$
$0.75=2-\frac{5}{4}$
$\frac{1}{2}+\frac{1}{2}=\frac{2}{3}\times\frac{3}{2}$
$\frac{1}{2}\times\frac{1}{2}=1-\frac{3}{4}$
$0.1=\frac{3}{5}-\frac{1}{2}$
$\frac{1}{4}+\frac{1}{4}=0.5$

Erase the labels for dots before continuing with the next exercise.

Exercise 2

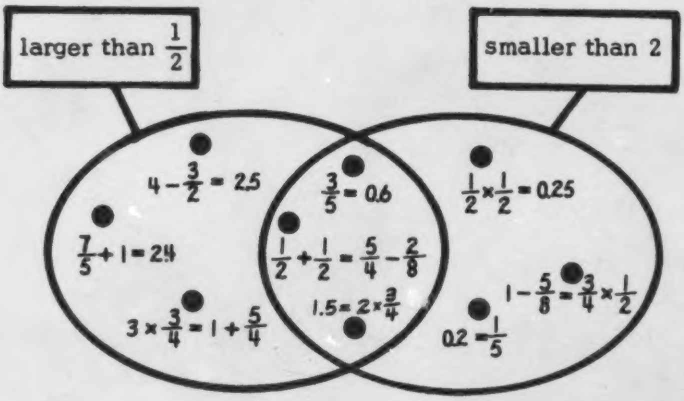
Divide the class into two teams. Add to your picture on the board as shown in the next illustration and assign number labels to the two teams by listing them on two sides of a Team Board.

| Team A | Team B |
|---|---|
| $\frac{1}{2} \times \frac{1}{2}$ | $0.25$ |
| $\frac{7}{5} + 1$ | $2 \times \frac{3}{4}$ |
| $0.2$ | $1 + \frac{5}{4}$ |
| $\frac{1}{2} + \frac{1}{2}$ | $2.4$ |
| $1 - \frac{5}{8}$ | $\frac{5}{4} - \frac{2}{8}$ |
| $1.5$ | $0.6$ |
| $3 \times \frac{3}{4}$ | $\frac{1}{5}$ |
| $\frac{3}{5}$ | $\frac{3}{4} \times \frac{1}{2}$ |
| $2.5$ | $4 - \frac{3}{2}$ |

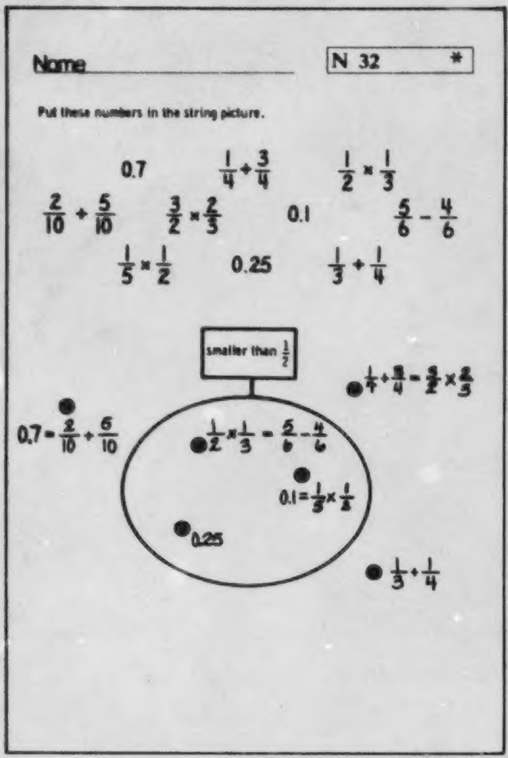

larger than $\frac{1}{2}$
smaller than 2

T: We are going to play a game. The teams will take turns trying to label correctly the dots in the string picture. You may only use the labels on your side of the Team Board. The team that correctly uses all its labels first is the winner.

Alternate teams and among the members of a team to call on students to label the dots. When a label is correct (the number is correctly placed in the string picture), erase it from the Team Board. At the conclusion of the game your string picture should look like the one in the next illustration.


larger than $\frac{1}{2}$
smaller than 2
$4 - \frac{3}{2} = 2.5$
$\frac{7}{5} + 1 = 2.4$
$3 \times \frac{3}{4} = 1 + \frac{5}{4}$
$\frac{3}{5} = 0.6$
$\frac{1}{2} + \frac{1}{2} = \frac{5}{4} - \frac{2}{8}$
$1.5 = 2 \times \frac{3}{4}$
$\frac{1}{2} \times \frac{1}{2} = 0.25$
$1 - \frac{5}{8} = \frac{3}{4} \times \frac{1}{2}$
$0.2 = \frac{1}{5}$

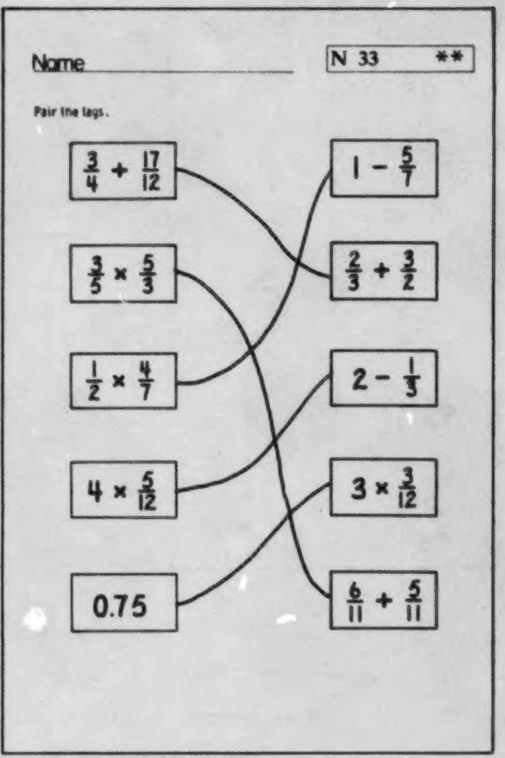
Worksheets N32, N33, and N34 are available for independent work.



0.7
$\frac{1}{4}+\frac{3}{4}$
$\frac{1}{2}\times\frac{1}{3}$
$\frac{2}{10}+\frac{5}{10}$
$\frac{3}{2}\times\frac{2}{3}$
0.1
$\frac{5}{6}-\frac{4}{6}$
$\frac{1}{5}\times\frac{1}{2}$
0.25
$\frac{1}{3}+\frac{1}{4}$
smaller than $\frac{1}{2}$
$\frac{1}{4}+\frac{3}{4}=\frac{3}{2}\times\frac{2}{3}$
$0.7=\frac{2}{10}+\frac{6}{10}$
$\frac{1}{2}\times\frac{1}{3}=\frac{5}{6}-\frac{4}{6}$
$0.1=\frac{1}{5}\times\frac{1}{2}$
0.25
$\frac{1}{3}+\frac{1}{4}$

Name ____________ N 32 *

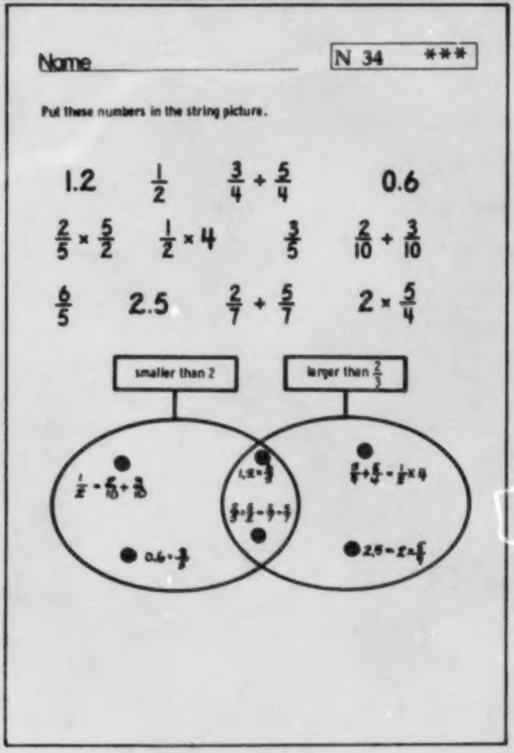
Put these numbers in the string picture.



$\frac{3}{4}+\frac{17}{12}$
$\frac{3}{5}\times\frac{5}{3}$
$\frac{1}{2}\times\frac{4}{7}$
$4\times\frac{5}{12}$
0.75
$1-\frac{5}{7}$
$\frac{2}{3}+\frac{3}{2}$
$2-\frac{1}{3}$
$3\times\frac{3}{12}$
$\frac{6}{11}+\frac{5}{11}$

Name ____________ N 33 **

Pair the tags.



1.2
$\frac{1}{2}$
$\frac{3}{4}+\frac{5}{4}$
0.6
$\frac{2}{5}\times\frac{5}{2}$
$\frac{1}{2}\times 4$
$\frac{3}{5}$
$\frac{2}{10}+\frac{3}{10}$
$\frac{6}{5}$
2.5
$\frac{2}{7}+\frac{5}{7}$
$2\times\frac{5}{4}$
smaller than 2
larger than $\frac{2}{3}$
$\frac{1}{2}=\frac{2}{10}+\frac{3}{10}$
$0.6=\frac{3}{5}$
$1.2=\frac{6}{5}$
$\frac{2}{5}\times\frac{5}{2}=\frac{2}{7}+\frac{5}{7}$
$\frac{3}{4}+\frac{5}{4}=\frac{1}{2}\times 4$
$2.5=2\times\frac{5}{4}$

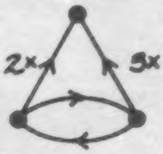
Name ____________ N 34 ***

Put these numbers in the string picture.

N 33

## 2.9 Division of Rational Numbers

CAPSULE LESSON SUMMARY



Label arrows in an arrow picture such as : Observe that the red arrows are return arrows of each other and find two ways to label each, where one way is of the form "÷□". Use such pictures to help do calculations involving division by rational numbers (expressed as fractions); for example,

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}.$$

In the context of an experiment of randomly picking a piece of paper, marked with a numeral or an operation symbol, from each of three boxes, do some multiplication, division, and addition calculations with fractions. Determine the probability of getting a number from $\frac{1}{3}$ to $\frac{5}{4}$ (inclusive).

MATERIALS

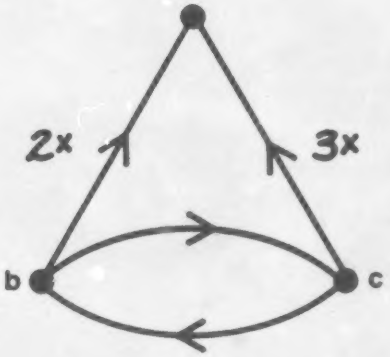
Teacher : Colored chalk

Student : Paper; colored pencils; Worksheet N35

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



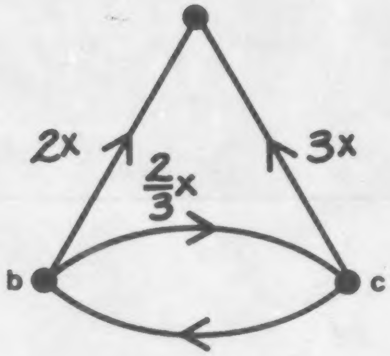
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

# N 33

T (tracing the arrow from b to c): What could this red arrow be for?

S: $\frac{2}{3}$x, because 2x followed by "against" 3x is $\frac{2}{3}$x.

Label the arrow "$\frac{2}{3}$x".



T (tracing the arrow from c to b): What could this red arrow be for?

S: $\frac{3}{2}$x, because 3x followed by "against" 2x is $\frac{3}{2}$x.

Label the arrow "$\frac{3}{2}$x".

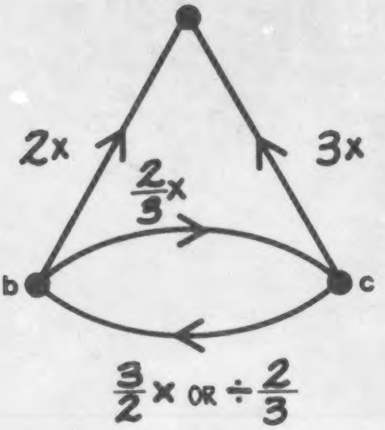
T: Good. Does the picture suggest another way to label this arrow?

Give the students a minute to examine the picture.

S: It's the return arrow for $\frac{2}{3}$x so is for $\div\frac{2}{3}$.

If "$\div\frac{2}{3}$" is not suggested, prompt the students to observe that it is a return arrow for an already labeled arrow and suggest labeling it "$\div\square$".

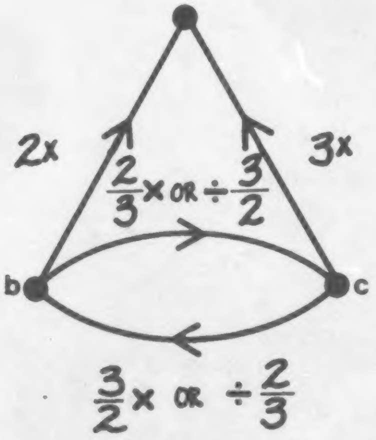
Label the arrow "$\div \frac{2}{3}$".


2x
3x
$\frac{2}{3}$x
b
c
$\frac{3}{2}$x OR $\div\frac{2}{3}$

T: Could we label the $\frac{2}{3}$x arrow in another way?

S: $\div\frac{3}{2}$, because it's the return arrow for $\frac{3}{2}$x

Label the arrow "$\div\frac{3}{2}$".


2x
3x
$\frac{2}{3}$x OR $\div\frac{3}{2}$
b
c
$\frac{3}{2}$x OR $\div\frac{2}{3}$

T (tracing the appropriate arrows): Dividing by $\frac{3}{2}$ is the same as multiplying by $\frac{2}{3}$. Likewise, dividing by $\frac{2}{3}$ is the same as multiplying by $\frac{3}{2}$.

# N 33

Label b "30" and write these problems on the board.

$$\frac{2}{3} \times 30 =$$

$$30 \div \frac{3}{2} =$$

T: Use the arrow picture to help solve these problems.

Allow a few minutes for individual work.

S: $\frac{2}{3} \times 30 = 20$.

S: They're the same problem; $30 \div \frac{3}{2} = \frac{2}{3} \times 30 = 20$.

Summarize this information on the board.

$$\frac{2}{3} \times 30 = 20$$

$$30 \div \frac{3}{2} = \frac{2}{3} \times 30 = 20$$

Write these two problems on the board.

$$\frac{1}{2} \div \frac{3}{2} =$$

$$\frac{3}{4} \div \frac{2}{3} =$$

T: Use the arrow picture to help solve these problems.

After a minute or two, invite students to explain their solutions.

S (pointing to b and tracing the $\div\frac{3}{2}$ arrow) : Put $\frac{1}{2}$ here (at b) in the picture; $\frac{1}{2} \div \frac{3}{2}$ is the same as $\frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$ or $\frac{1}{3}$.

S (pointing to c and tracing the $\div\frac{2}{3}$ arrow) : Put $\frac{3}{4}$ here (at c); $\frac{3}{4} \div \frac{2}{3}$ is the same as $\frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$.

Write these problems on the board and ask the students to solve them. (Answers are in the boxes.)

$$12 \div \frac{2}{3} = \boxed{12 \times \frac{3}{2} = 18}$$

$$\frac{4}{5} \div \frac{5}{6} = \boxed{\frac{4}{5} \times \frac{6}{5} = \frac{24}{25}}$$

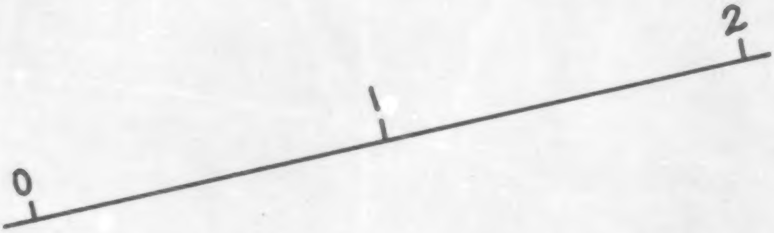
$$\frac{1}{3} \div \frac{3}{4} = \boxed{\frac{1}{3} \times \frac{4}{3} = \frac{4}{9}}$$

$$6 \div \frac{8}{5} = \boxed{6 \times \frac{5}{8} = \frac{30}{8} = \frac{15}{4} = 3\frac{3}{4}}$$

Allow several minutes for individual work. Encourage those students who have difficulty to draw an appropriate arrow picture. Check the work collectively by asking students to give explanations at the board.

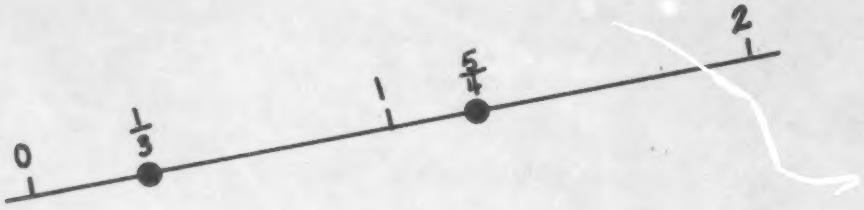
Exercise 2

Draw this part of the number line on the board.

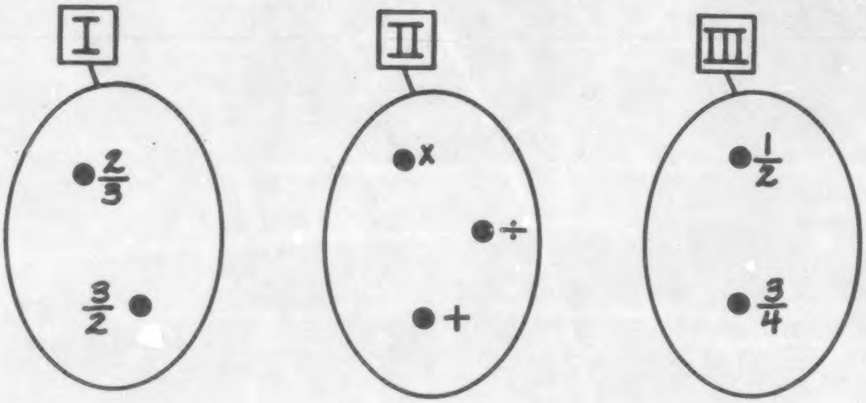


# N 33

Invite students to locate $\frac{1}{3}$ and $\frac{5}{4}$ approximately; then add this green segment to your drawing.



Draw these three strings on the board to the side of the number line.



T: Imagine three boxes with some little pieces of paper. Each piece of paper has a symbol written on it and folded so that the symbol is hidden. The string picture shows what is in the boxes.

Suppose we pick one piece of paper from each box. Give me a number we could get from the first box?

S: $\frac{2}{3}$.

Continue in a similar manner for Box II and Box III. Write the symbols on the board as the students announce what you could get. You will adapt the discussion to the results in your classroom, but suppose that $\frac{2}{3}$, x, and $\frac{3}{4}$ are given.

# $\frac{2}{3} \times \frac{3}{4}$

T: We get a number. Which number? Calculate "$\frac{2}{3} \times \frac{3}{4}$".

S: $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$.

T (pointing to the number line): Does $\frac{1}{2}$ belong to the green segment?

S: Yes, $\frac{1}{2}$ is larger than $\frac{1}{3}$ and smaller than $\frac{5}{4}$.

Ask a student to point to where $\frac{1}{2}$ is located on the number line.

Repeat this activity once more. Perhaps $\frac{3}{2}$, +, and $\frac{1}{2}$ will be given this time, in which case you get $\frac{3}{2} + \frac{1}{2} = 2$ and 2 does not belong to the green segment.

T: We have seen a number that belongs to the green segment and one that does not belong to the green segment. If we randomly choose a paper from each box, what is the probability of getting a number that belongs to the green segment?

Let the class discuss this question. Someone should suggest that you try to find all the possible numbers. Distribute Worksheet N35 and instruct the students to calculate all the possible numbers. While the students are working, draw the three tables from the worksheet on the board.

After a short while, ask several students to complete the tables on the board. See the answer key on page 101. Be sure to discuss collectively any answers that are disputed.

T: Now what must we do?

S: Decide which numbers in the tables belong to the green segment.

T: On your worksheet, circle in green all the numbers that belong to the green segment.

After a few minutes, invite students to circle the numbers in the tables on the board. Be prepared to discuss any numbers the students question.

T: We circled six of the numbers. What is the probability of getting a number that belongs to the green segment?

S: $\frac{6}{12}$ or $\frac{1}{2}$.

S: One chance out of two.

Name ______________________

N 35

Complete the tables.

| $\times$ | $\frac{1}{2}$ | $\frac{3}{4}$ |
|---|---|---|
| $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{2}$ |
| $\frac{3}{2}$ | $\frac{3}{4}$ | $1\frac{1}{8}$ |

| $+$ | $\frac{1}{2}$ | $\frac{3}{4}$ |
|---|---|---|
| $\frac{2}{3}$ | $1\frac{1}{3}$ | $\frac{8}{9}$ |
| $\frac{3}{2}$ | 3 | 2 |

| $+$ | $\frac{1}{2}$ | $\frac{3}{4}$ |
|---|---|---|
| $\frac{2}{3}$ | $1\frac{1}{6}$ | $1\frac{5}{12}$ |
| $\frac{3}{2}$ | 2 | $2\frac{1}{4}$ |

(Other correct names could be given; e.g., $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \ldots$ .)

# 3

# HAND-CALCULATOR PUZZLES

In this chapter, the lessons exploit the hand-calculator to present situations involving mental arithmetic. Basically, students explore various ways to display a given number on a "special" hand-calculator, that is, a hand-calculator with only two number keys (such as, $\boxed{2}$ and $\boxed{8}$ , or $\boxed{5}$ and $\boxed{8}$ , or $\boxed{4}$ and $\boxed{6}$ ) and any other key except $\boxed{\cdot}$ . An additional challenge is to find a solution pressing as few keys as possible.

CAPSULE LESSON SUMMARIES

Section 3.1: Allow just the number keys $\boxed{2}$ and $\boxed{8}$ and any other key except $\boxed{\cdot}$ in putting numbers on the display of a hand-calculator. Find solutions which require pressing keys at most twelve times. Build arrow roads between two given numbers where the arrows must be for +, −, x, or ÷ some number whose digits are 2 or 8. Try to find shortest such arrow roads.

Section 3.2: Allow just the number keys $\boxed{4}$ and $\boxed{6}$ and any other key except $\boxed{\cdot}$ in putting numbers on the display of a hand-calculator. Find solutions which require pressing keys at most ten times. Build arrow roads between two given numbers where the arrows must be for +, −, x, or ÷ some number whose digits are 4 or 6. Try to find shortest such arrow roads.

Section 3.3: Allow just the number keys $\boxed{5}$ and $\boxed{8}$ and any other key except $\boxed{\cdot}$ in putting numbers on the display of a hand-calculator. Find solutions which require pressing keys at most ten times. Build arrow roads between two given numbers where the arrows must be for +, −, x, or ÷ some number whose digits are 5 or 8. Try to find shortest such arrow roads.

# N 4

## 3.1 Hand-Calculator Puzzles #1

CAPSULE LESSON SUMMARY

Allow just the number keys $\boxed{2}$ and $\boxed{8}$ and any other key except $\boxed{\cdot}$ in putting numbers on the display of a hand-calculator. Find solutions which require pressing keys at most twelve times. Build arrow roads between two given numbers where the arrows must be for +, -, x, or ÷ by some number whose digits are 2 or 8. Try to find shortest such arrow roads.

MATERIALS

Teacher: Hand-calculator; colored chalk
Student: Hand-calculator; paper; colored pencils

### DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$\boxed{2} \quad \boxed{8} \quad \xcancel{\boxed{\cdot}}$$

T: The only number keys you may press are $\boxed{2}$ and $\boxed{8}$. You may use any other keys except $\boxed{\cdot}$. So, for example, you can add, subtract, multiply or divide by any numbers whose digits are 2 or 8. Try to put 3 on the display of your calculator.

Give the class a few minutes to explore this situation. Walk around the room observing the individual work and encouraging those who need it.

There are many solutions to this problem. Since we cannot predict which solutions your students will find, we will present some of the solutions discovered by students in our development classes.

S: Press $\boxed{8}\ \boxed{-}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$.

Write the student's solution on the board.

$$3:\ \boxed{8}\ \boxed{-}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$$

T: Are there other solutions?

Accept several more solutions before continuing with other numbers. Sometimes ask the class to check a solution by pressing the suggested sequence of keys on the hand-calculator. It is not necessary to verify every solution. Some other solutions are presented below.[*]

$$3:\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{=}$$

$$\boxed{2}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$$

$$\boxed{8}\ \boxed{+}\ \boxed{2}\ \boxed{=}\ \boxed{=}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{=}$$

$$\boxed{2}\ \boxed{\times}\ \boxed{8}\ \boxed{+}\ \boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$$

Continue this activity with other numbers such as 5, 7, 9, 25, 0.5, and 8.2. Challenge the class to think about shorter solutions by introducing the condition that it costs one cent to press a key and asking for solutions that cost no more than twelve cents. Also, permit students to experiment with "unknown" keys if they wish. For example, a solution for 3 could be $\boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{+}\ \boxed{8}\ \boxed{=}\ \boxed{\sqrt{\ }}$.

---

[*] Throughout this chapter we assume that hand-calculators with an automatic constant feature are used. For a more detailed discussion of hand-calculators, see the General Introduction Section 2(f) on Role and Use of Hand-calculators.

# N 4

You may want to encourage some students to find solutions using as many different operations as possible (even if more than twelve cents is spent). Several solutions to each problem are provided here.

5: $\boxed{8}\ \boxed{+}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$ 6¢

$\boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{=}\ \boxed{=}$ 7¢

$\boxed{8}\ \boxed{\times}\ \boxed{=}\ \boxed{\div}\ \boxed{2}\ \boxed{+}\ \boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$ 10¢

7: $\boxed{2}\ \boxed{8}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{=}$ 6¢

$\boxed{2}\ \boxed{8}\ \boxed{\times}\ \boxed{2}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$ 7¢

$\boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{+}\ \boxed{2}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ 8¢

9: $\boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{+}\ \boxed{8}\ \boxed{=}$ 6¢

$\boxed{8}\ \boxed{+}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{=}\ \boxed{=}$ 9¢

$\boxed{8}\ \boxed{\times}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$ 8¢

25: $\boxed{8}\ \boxed{+}\ \boxed{2}\ \boxed{\times}\ \boxed{=}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{=}$ 9¢

$\boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{+}\ \boxed{8}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ 8¢

$\boxed{8}\ \boxed{\times}\ \boxed{8}\ \boxed{-}\ \boxed{8}\ \boxed{=}\ \boxed{=}\ \boxed{+}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$

12¢

0.5: $\boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{=}$ 5¢

$\boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$ 6¢

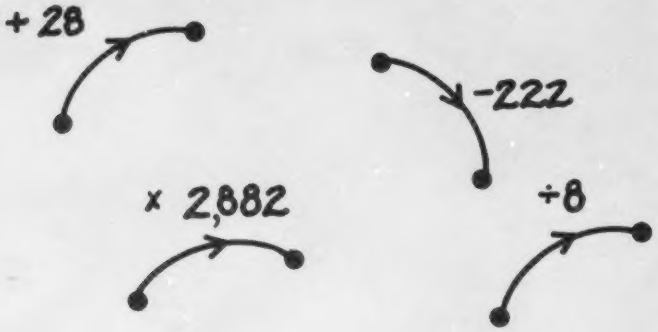
$\boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$ 6¢

8.2: $\boxed{8}\ \boxed{+}\ \boxed{2}\ \boxed{\div}\ \boxed{=}\ \boxed{=}\ \boxed{\times}\ \boxed{8}\ \boxed{2}\ \boxed{=}$ 10¢

$\boxed{2}\ \boxed{+}\ \boxed{8}\ \boxed{\div}\ \boxed{=}\ \boxed{=}\ \boxed{\times}\ \boxed{2}\ \boxed{+}\ \boxed{8}\ \boxed{=}$ 11¢

Exercise 2

Write this information on the board.

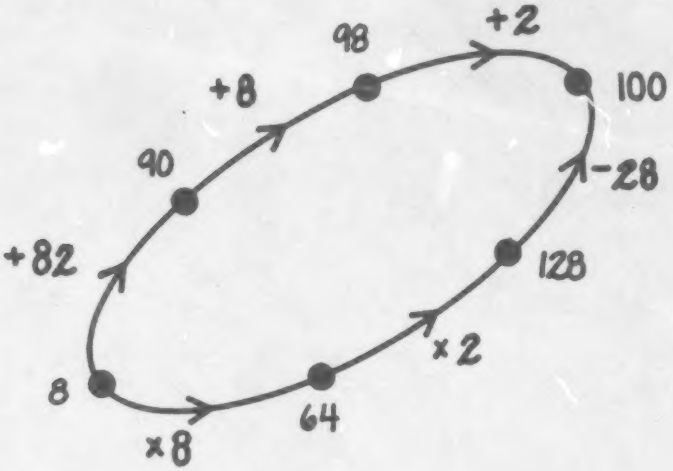


T: Here, red arrows can be for add, subtract, multiply, or divide by some number whose digits are 2 or 8. A few examples of what red arrows could be for are shown on the board. Build an arrow road from 8 to 100 using only these kinds of arrows. Try to find a shortest road.

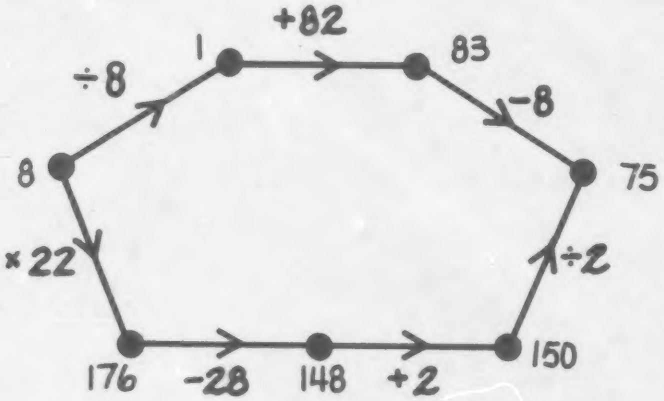
Let the students work independently on this problem. Encourage the faster workers to find more than one solution. When most students have found at least one solution, send volunteers to draw their roads on the board.

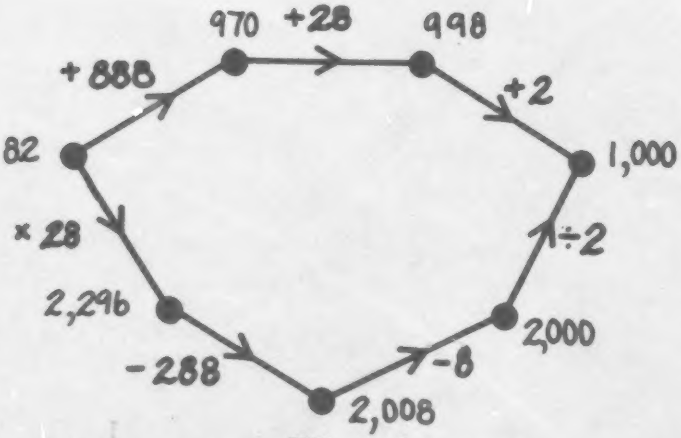
## N 4

There are many possible solutions, two of which are given below.


98
+2
+8
100
90
-28
+82
128
×2
8
64
×8

Continue this activity by asking students to draw other roads, for example, from 8 to 75, from 82 to 1,000, or from 28 to 0.25. Some solutions are given below and on the next page.


+82
1
83
÷8
-8
8
75
×22
÷2
176
-28
148
+2
150


970
+28
998
+888
+2
82
1,000
×28
÷2
2,296
2,000
-288
-8
2,008

**N 4**

$$\begin{tikzcd}28 \arrow[r, "\div 28"] & 1 \arrow[r, "\div 2"] & 0.5 \arrow[r, "\div 2"] & 0.25\end{tikzcd}$$

# N 12

## 3.2 Hand-Calculator Puzzles #2

CAPSULE LESSON SUMMARY

Allow just the number keys $\boxed{4}$ and $\boxed{6}$ and any other key except $\boxed{\cdot}$ in putting numbers on the display of a hand-calculator. Find solutions which require pressing keys at most ten times. Build arrow roads between two given numbers where the arrows must be for +, −, ×, or ÷ some number whose digits are 4 or 6. Try to find shortest such arrow roads.

MATERIALS

Teacher: Hand-calculator; colored chalk
Student: Hand-calculator; paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$\boxed{4}\ \boxed{6}\ \xcancel{\boxed{\cdot}}$$

T: The only number keys you may press are $\boxed{4}$ and $\boxed{6}$. You may use any other keys except $\boxed{\cdot}$. So, for example, you can add, subtract, multiply, or divide by any number whose digits are 4 or 6. Try to put 5 on the display of your calculator.

Give the class a few minutes to explore this situation. There are many solutions to this problem; here, we present some of those discovered by students in our development classes.

S: Press $\boxed{6}\ \boxed{\div}\ \boxed{6}\ \boxed{+}\ \boxed{4}\ \boxed{=}$.

Write the student's solution on the board.

$$5:\ \boxed{6}\ \boxed{\div}\ \boxed{6}\ \boxed{+}\ \boxed{4}\ \boxed{=}$$

T: Try to find some other solutions.

Accept several more solutions before continuing with other numbers. Sometimes ask the class to check a solution by pressing the suggested sequence of keys on the hand-calculator. It is not necessary to verify every solution. Some other solutions are presented below.

5: $\boxed{4}\ \boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{6}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$

$\boxed{6}\ \boxed{4}\ \boxed{-}\ \boxed{4}\ \boxed{4}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$

$\boxed{4}\ \boxed{+}\ \boxed{6}\ \boxed{\div}\ \boxed{4}\ \boxed{+}\ \boxed{=}$

$\boxed{4}\ \boxed{\div}\ \boxed{=}\ \boxed{+}\ \boxed{4}\ \boxed{=}$

T: Now display 7 on your calculator, but imagine that each time you press a key, it costs one cent. Try not to spend more than ten cents.

Several solutions and their prices are given below.

7: $\boxed{4}\ \boxed{\div}\ \boxed{4}\ \boxed{+}\ \boxed{6}\ \boxed{=}$ 6¢

$\boxed{4}\ \boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{4}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$ 8¢

$\boxed{6}\ \boxed{6}\ \boxed{\div}\ \boxed{6}\ \boxed{-}\ \boxed{4}\ \boxed{=}$ 7¢

$\boxed{6}\ \boxed{\div}\ \boxed{=}\ \boxed{+}\ \boxed{6}\ \boxed{=}$ 6¢

# N 12

Continue this activity with other numbers such as 23, 99, 0.5, and 0.1. Feel free to adjust your choice of numbers to the numerical abilities of your class. For example, weaker students might try to display 9 and 15 while stronger students should enjoy trying to display 0.64 and 0.9. Encourage your students to find several solutions for each number. Accept solutions that cost more than ten cents, but reinforce finding shorter solutions. Various solutions to these problems are presented below and on the next page.

| | | |
|---|---|---|
| 23: | $\boxed{6}\boxed{6}\boxed{\div}\boxed{6}\boxed{+}\boxed{6}\boxed{=}\boxed{=}$ | 8¢ |
| | $\boxed{4}\boxed{6}\boxed{+}\boxed{=}\boxed{\div}\boxed{4}\boxed{=}$ | 7¢ |
| | $\boxed{4}\boxed{6}\boxed{\div}\boxed{4}\boxed{+}\boxed{=}$ | 6¢ |
| | $\boxed{4}\boxed{6}\boxed{\div}\boxed{4}\boxed{\sqrt{\ }}\boxed{=}$ | 6¢ |
| 99: | $\boxed{4}\boxed{4}\boxed{\div}\boxed{4}\boxed{+}\boxed{4}\boxed{4}\boxed{=}\boxed{=}$ | 9¢ |
| | $\boxed{6}\boxed{6}\boxed{\div}\boxed{4}\boxed{\times}\boxed{6}\boxed{=}$ | 7¢ |
| | $\boxed{6}\boxed{\div}\boxed{6}\boxed{+}\boxed{6}\boxed{6}\boxed{+}\boxed{4}\boxed{4}\boxed{-}\boxed{6}\boxed{=}\boxed{=}$ | 13¢ |
| | $\boxed{4}\boxed{6}\boxed{+}\boxed{=}\boxed{M+}\boxed{6}\boxed{\div}\boxed{6}\boxed{+}\boxed{6}\boxed{+}\boxed{MRC}\boxed{=}$ | 13¢ |
| 0.5: | $\boxed{6}\boxed{-}\boxed{4}\boxed{\div}\boxed{4}\boxed{=}$ | 6¢ |
| | $\boxed{4}\boxed{+}\boxed{=}\boxed{\div}\boxed{4}\boxed{=}\boxed{=}$ | 7¢ |
| | $\boxed{4}\boxed{\div}\boxed{4}\boxed{=}\boxed{M+}\boxed{6}\boxed{\div}\boxed{4}\boxed{-}\boxed{MRC}\boxed{=}$ | 11¢ |
| | $\boxed{4}\boxed{\sqrt{\ }}\boxed{\div}\boxed{=}\boxed{=}$ | 5¢ |

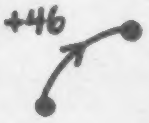
**N 12**

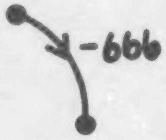
| | | |
|---|---|---|
| 0.1: | $\boxed{6}\boxed{+}\boxed{4}\boxed{\div}\boxed{=}\boxed{=}$ | 6¢ |
| | $\boxed{6}\boxed{\times}\boxed{6}\boxed{+}\boxed{4}\boxed{=}\boxed{M_+}\boxed{4}\boxed{\div}\boxed{\text{MRC}}\boxed{=}$ | 11¢ |
| | $\boxed{4}\boxed{\times}\boxed{=}\boxed{=}\boxed{-}\boxed{4}\boxed{=}\boxed{M_+}\boxed{6}\boxed{\div}\boxed{\text{MRC}}\boxed{=}$ | 12¢ |
| | $\boxed{6}\boxed{+}\boxed{4}\boxed{\div}\boxed{=}\boxed{=}\boxed{=}\boxed{\sqrt{\ }}$ | 8¢ |
| 9: | $\boxed{6}\boxed{\times}\boxed{6}\boxed{+}\boxed{4}\boxed{=}$ | 6¢ |
| | $\boxed{4}\boxed{4}\boxed{+}\boxed{6}\boxed{+}\boxed{4}\boxed{\div}\boxed{6}\boxed{=}$ | 9¢ |
| | $\boxed{4}\boxed{\div}\boxed{4}\boxed{+}\boxed{4}\boxed{=}\boxed{=}$ | 7¢ |
| | $\boxed{4}\boxed{6}\boxed{+}\boxed{4}\boxed{=}\boxed{=}\boxed{\div}\boxed{6}\boxed{=}$ | 9¢ |
| 15: | $\boxed{4}\boxed{4}\boxed{\div}\boxed{4}\boxed{+}\boxed{4}\boxed{=}$ | 7¢ |
| | $\boxed{6}\boxed{4}\boxed{-}\boxed{4}\boxed{\div}\boxed{4}\boxed{=}$ | 7¢ |
| | $\boxed{6}\boxed{\times}\boxed{6}\boxed{\div}\boxed{4}\boxed{+}\boxed{6}\boxed{=}$ | 8¢ |
| | $\boxed{4}\boxed{\div}\boxed{4}\boxed{+}\boxed{4}\boxed{+}\boxed{=}\boxed{=}$ | 8¢ |
| 0.9: | $\boxed{6}\boxed{+}\boxed{4}\boxed{\div}\boxed{=}\boxed{=}\boxed{M_+}\boxed{4}\boxed{\div}\boxed{4}\boxed{-}\boxed{\text{MRC}}\boxed{=}$ | 13¢ |
| 0.64: | $\boxed{6}\boxed{+}\boxed{4}\boxed{\div}\boxed{=}\boxed{=}\boxed{=}\boxed{\times}\boxed{6}\boxed{4}\boxed{=}$ | 11¢ |

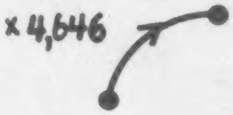
# N 12

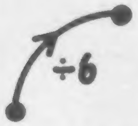
## Exercise 2

Write this information on the board.



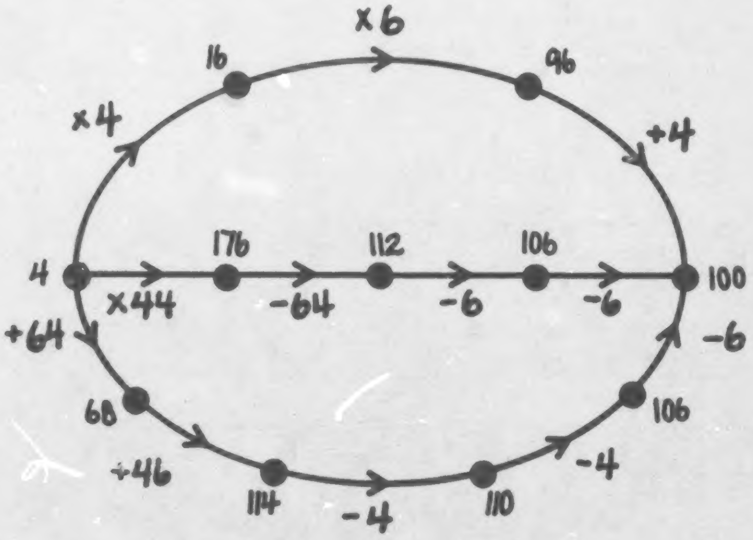




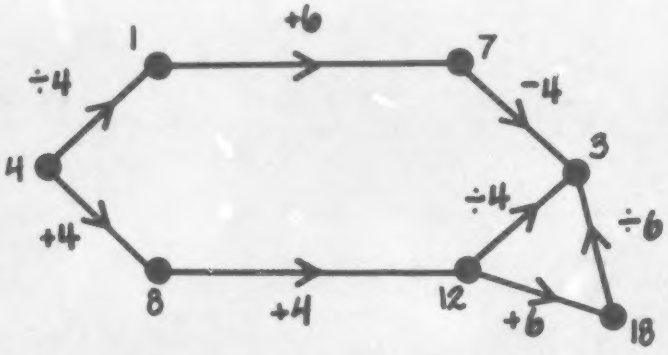


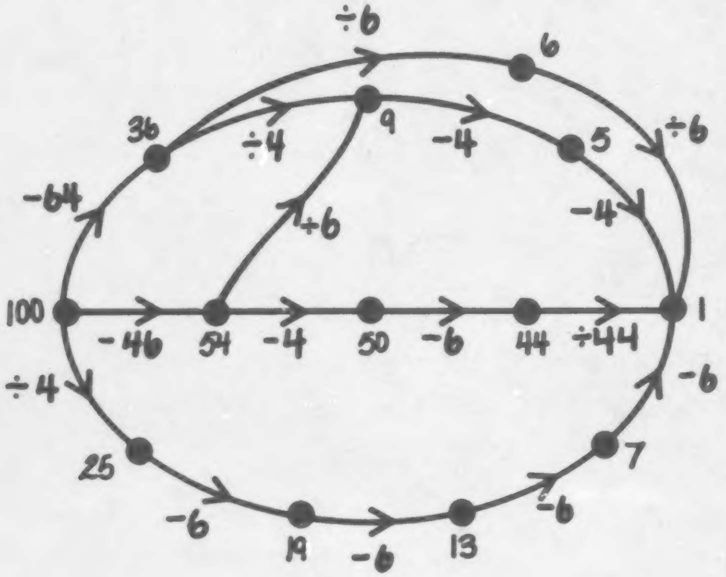
T: Here, the red arrows can be for add, subtract, multiply, or divide by some number whose digits are 4 or 6. A few examples of what red arrows could be for are shown on the board. Build an arrow road from 4 to 100 using only these kinds of arrows. Try to find a shortest road.

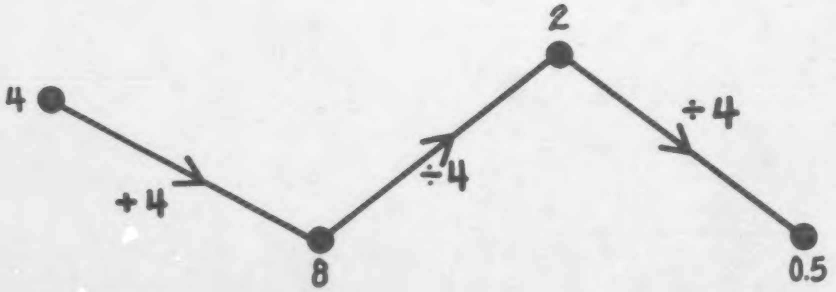
Let the students work independently on this problem. Encourage the faster workers to find more than one solution. When most students have found at least one solution, send volunteers to draw their roads on the board. There are many possible solutions, three of which are given below.



Continue this activity by inviting students to draw roads from 4 to 3, from 100 to 1, or from 4 to 0.5. Some solutions are given below.


+6
1
7
÷4
-4
4
3
÷4
÷6
+4
8
+4
12
+6
18


÷6
6
36
÷4
9
-4
5
÷6
-64
-4
÷6
100
1
-46
54
-4
50
-6
44
÷44
÷4
-6
25
7
-6
19
-6
13
-6



# N 32

## 3.3 Hand-Calculator Puzzles #3

CAPSULE LESSON SUMMARY

Allow just the number keys $\boxed{5}$ and $\boxed{8}$ and any other key except $\boxed{\cdot}$ in putting numbers on the display of a hand-calculator. Find solutions which require pressing keys at most ten times. Build arrow roads between two given numbers where the arrows must be for +, −, x, or ÷ some number whose digits are 5 or 8. Try to find shortest such arrow roads.

MATERIALS

Teacher: Hand-calculator; colored chalk
Student: Hand-calculator; paper; colored pencils

### DESCRIPTION OF LESSON

#### Exercise 1

Write this information on the board.

$$\boxed{5}\ \boxed{8}\ \xcancel{\boxed{\cdot}}$$

T: The only number keys you may press are $\boxed{5}$ and $\boxed{8}$. You may use any other keys except $\boxed{\cdot}$. So, for example, you can add, subtract, multiply, or divide by any number whose digits are 5 or 8. Try to put 7 on the display of your calculator.

Give the class a few minutes to explore this situation. There are many solutions to this problem; here, we present some of those discovered by students in our development classes.

S: Press $\boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{-}\ \boxed{8}\ \boxed{=}$.

Write the student's solution on the board.

$$7\colon \boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{-}\ \boxed{8}\ \boxed{=}$$

**T :** Try to find some other solutions.

Accept several more solutions before continuing with other numbers. Sometimes ask the class to check a solution by pressing the suggested sequence of keys or the hand-calculator. It is not necessary to verify each solution. Some other solutions are presented below.

$$7: \boxed{5}\,\boxed{\times}\,\boxed{8}\,\boxed{-}\,\boxed{5}\,\boxed{\div}\,\boxed{5}\,\boxed{=}$$

$$\boxed{8}\,\boxed{\times}\,\boxed{8}\,\boxed{-}\,\boxed{8}\,\boxed{\div}\,\boxed{8}\,\boxed{=}$$

$$\boxed{5}\,\boxed{+}\,\boxed{=}\,\boxed{=}\,\boxed{-}\,\boxed{8}\,\boxed{=}$$

$$\boxed{8}\,\boxed{+}\,\boxed{5}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{-}\,\boxed{8}\,\boxed{=}\,\boxed{=}$$

**T :** Now display 6 on your calculator, but imagine that each time you press a key, it costs one cent. Try not to spend more than ten cents.

Several solutions and their prices are given below.

$$6: \boxed{5}\,\boxed{5}\,\boxed{\div}\,\boxed{5}\,\boxed{-}\,\boxed{5}\,\boxed{=} \qquad 7¢$$

$$\boxed{5}\,\boxed{\times}\,\boxed{8}\,\boxed{+}\,\boxed{8}\,\boxed{\div}\,\boxed{8}\,\boxed{=} \qquad 8¢$$

$$\boxed{5}\,\boxed{\div}\,\boxed{=}\,\boxed{+}\,\boxed{5}\,\boxed{=} \qquad 6¢$$

$$\boxed{8}\,\boxed{-}\,\boxed{5}\,\boxed{+}\,\boxed{=} \qquad 5¢$$

# N 32

Continue this activity with other numbers such as 35, 100, 101, 0.5, and 0.2. Encourage your students to find several solutions for each number. Accept solutions that cost more than ten cents, but reinforce finding shorter solutions. Various solutions to these problems are presented below and on the next page.

35: $\boxed{5}\,\boxed{\times}\,\boxed{8}\,\boxed{-}\,\boxed{5}\,\boxed{=}$ 6¢

$\boxed{5}\,\boxed{+}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ 8¢

$\boxed{5}\,\boxed{5}\,\boxed{-}\,\boxed{5}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ 8¢

$\boxed{5}\,\boxed{\times}\,\boxed{=}\,\boxed{+}\,\boxed{5}\,\boxed{=}\,\boxed{=}$ 7¢

100: $\boxed{8}\,\boxed{5}\,\boxed{+}\,\boxed{5}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ 7¢

$\boxed{5}\,\boxed{5}\,\boxed{+}\,\boxed{=}\,\boxed{-}\,\boxed{5}\,\boxed{=}\,\boxed{=}$ 8¢

$\boxed{5}\,\boxed{\times}\,\boxed{=}\,\boxed{+}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ 7¢

$\boxed{5}\,\boxed{\times}\,\boxed{8}\,\boxed{+}\,\boxed{5}\,\boxed{+}\,\boxed{5}\,\boxed{5}\,\boxed{=}$ 9¢

101: $\boxed{8}\,\boxed{5}\,\boxed{+}\,\boxed{8}\,\boxed{=}\,\boxed{=}$ 6¢

$\boxed{5}\,\boxed{8}\,\boxed{+}\,\boxed{=}\,\boxed{-}\,\boxed{5}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ 9¢

$\boxed{8}\,\boxed{8}\,\boxed{+}\,\boxed{8}\,\boxed{+}\,\boxed{5}\,\boxed{=}$ 7¢

0.5: $\boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$ 9¢

$\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$ 10¢

$\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}$ 8¢

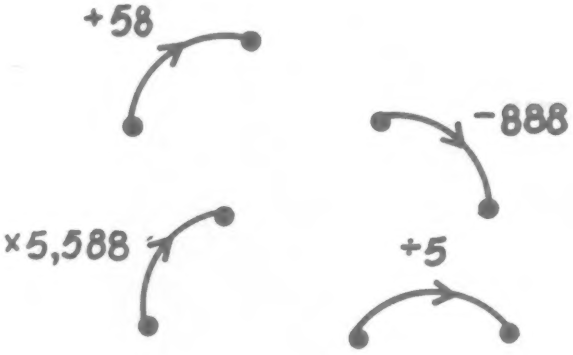
0.2: $\boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}$ 4¢

$\boxed{5}\ \boxed{\times}\ \boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{\div}\ \boxed{5}\ \boxed{=}\ \boxed{=}$ 9¢

$\boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$ 6¢

## Exercise 2

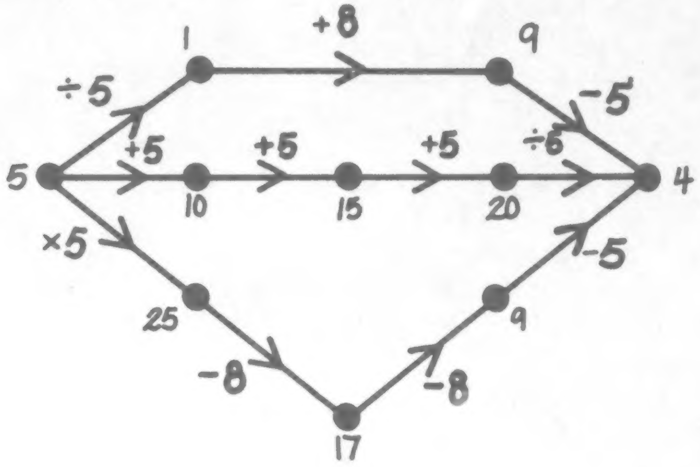
Write this information on the board.



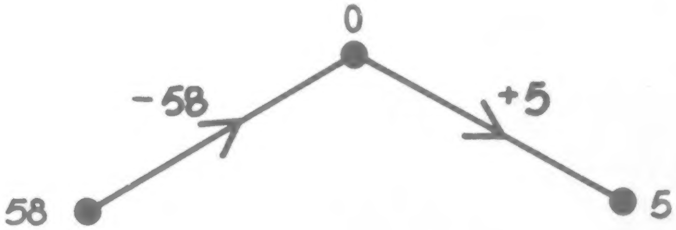
T: Here the red arrows can be for add, subtract, multiply, or divide by some number whose digits are 5 or 8. A few examples of what red arrows could be for are shown on the board. Build an arrow road from 5 to 4 using only these kinds of arrows. Try to find a shortest road.

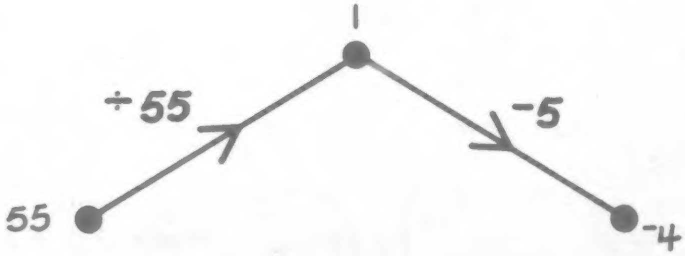
# N 32

Let the students work independently on this problem. Encourage the faster workers to find more than one solution. When most students have found at least one solution, send volunteers to draw their roads on the board. There are many possible solutions, a few of which are given below.

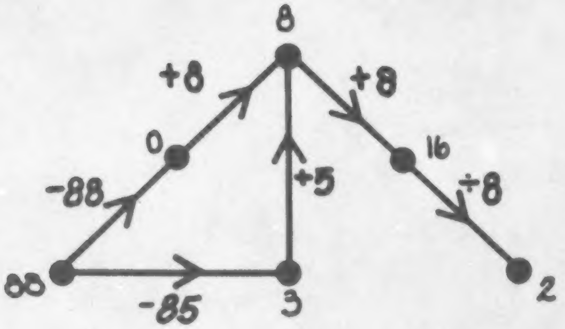


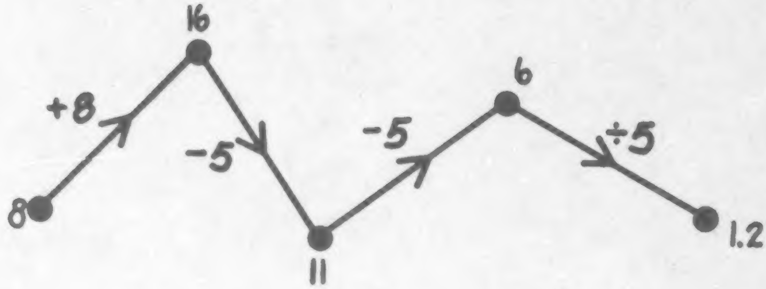
Continue this activity by inviting students to draw other roads, for example, from 58 to 5, from 55 to ⁻4, from 88 to 2, from 8 to 1.2, or from 8 to 7.5. Some solutions are given below and on the next page.

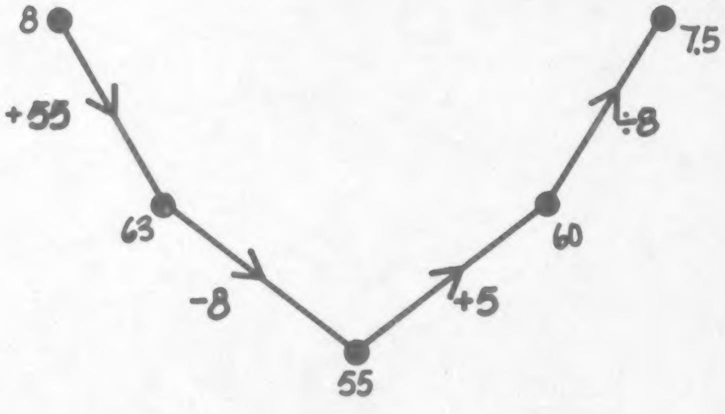




**N 32**


8
+8
+8
0
16
-88
+5
÷8
88
-85
3
2


16
+8
-5
-5
6
÷5
8
11
1.2


8
7.5
+55
÷8
63
60
-8
+5
55

# 4

# COMBINATORICS

"Combinatorics" is a term meaning "the process of (systematically) counting the number of possible ways or outcomes in certain situations." In this chapter, several situations are presented in which it is of interest to "count the possibilities." These situations include:

- counting how many different relations there are on a set of three objects;
- counting how many different relations or functions there are with arrows starting in a set A and ending in another set B;
- counting how many permutations there are on a set of six objects.

In each of these situations a procedure is found for "counting the possibilities." In some cases a problem of finding the probability that a certain kind of outcome will occur provides additional interest to the combinatorics situation.

## CAPSULE LESSON SUMMARIES

| | |
|---|---|
| Section 4.1: | Count the number of different relations on a set of three objects by using codes, in particular a binary code, and assigning code numbers to the relations.<br>(Worksheets N36, N37) |
| Section 4.2: | Count the number of different relations with arrows starting from dots in a set A (five members) and ending at dots in a set B (four members) by using codes, in particular a binary code, and assigning code numbers to the relations. |

Section 4.3: Review the notion of a function and count the number of different functions with arrows starting from dots in a set A (five members) and ending at dots in a set B (four members). Compare the number of functions to the number of relations (found in Section 4.2). Repeat the counting activity for both relations and functions with arrows from a new set A to a new set B.

Section 4.4: Play a game with six students resulting in a permutation (special kind of function with exactly one arrow starting and ending at each dot) on a set of six objects. Observe that the arrow pictures for permutations may have from one to six pieces. Count the number of permutations and then the number that have connected (one-piece) arrow pictures. Find that when playing the game, the probability of getting a connected arrow picture is $\frac{1}{6}$. (Worksheets N38, N39)

## 4.1 Counting Relations #1

CAPSULE LESSON SUMMARY

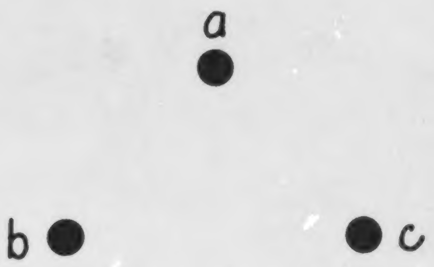
Count the number of different relations on a set of three objects by using codes, in particular a binary code, and assigning code numbers to the relations.

MATERIALS

Teacher: Colored chalk; magnetic checkers (optional)
Student: Paper; colored pencils; Worksheets N36 and N37

### DESCRIPTION OF LESSON

Draw this picture on the board.



T: Draw three dots on your paper and label them "a", "b", and "c". Then draw any arrow picture you wish with these three dots. When you have completed one picture, draw several more.

Allow several minutes for individual work. If necessary, indicate that it would be redundant to have two arrows start at the same dot and end at the same dot. After a while, invite several students to put their pictures on the board. Since we cannot predict which pictures your students will draw, the next illustration

# N 5

presents four pictures that students in our development classes drew.

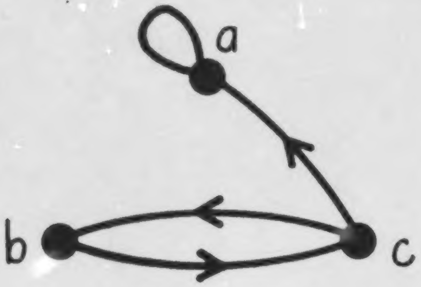


Figure 1

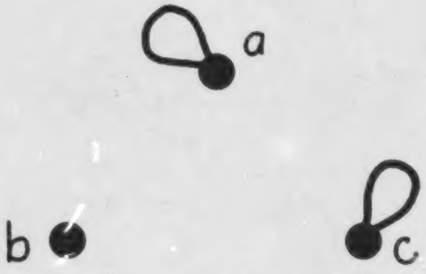


Figure 2

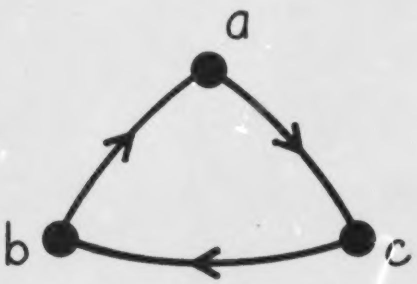


Figure 3

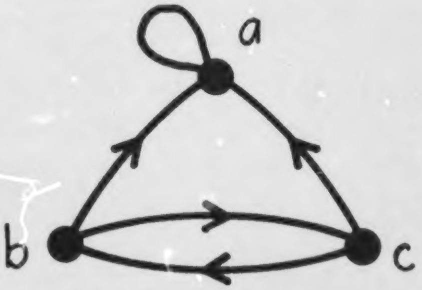


Figure 4

T: Each of these pictures is for a relation on a set of three objects. Which of these relations is a function?

S: Figures 2 and 3 are functions. Figure 1 has a guilty point at c and Figure 4 has guilty points at b and c.

T: Good. How many different relations do you think there are on a set of three objects?

Write some of the students' guesses on the board. At this point accept any guess; the rest of the lesson will be directed at finding the exact number of relations on a set of three objects.

Write this information on the board.

| | | | |
|---|---|---|---|
| c | | ● | |
| b | | | ● |
| a | ● | | ● |
| | a | b | c |

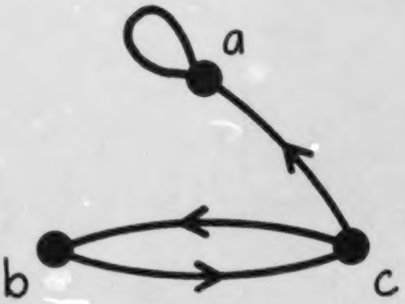


Figure 1

T: To help find the number of relations, Nabu invented two different codes. The first involves putting checkers[*] in a three-by-three grid. The code for Figure 1 is shown on the grid. Can you explain this code?

Give the class a few minutes to study the picture.

S: There is an arrow starting at c and ending at b in Figure 1. On the grid, there is a checker in the square above "c" and to the right of "b" (across to "c", then up to "b"). The same is true for each of the remaining three arrows.

T: That's correct. On your paper, code each of the other arrow pictures in a grid.

---

[*] If you do not have a magnetic board on which to put the grid, simply draw red dots for checkers.

## N 5

As the class completes this work, invite three students to put their grid pictures on the board.

| | | | |
|---|---|---|---|
| c | | | ● |
| b | | | |
| a | ● | | |
| | a | b | c |

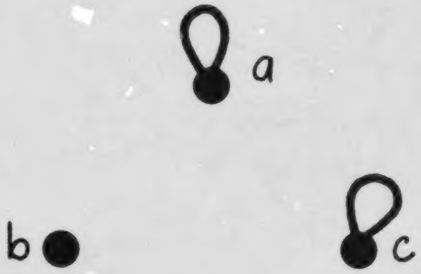


Figure 2

| | | | |
|---|---|---|---|
| c | ● | | |
| b | | | ● |
| a | | ● | |
| | a | b | c |

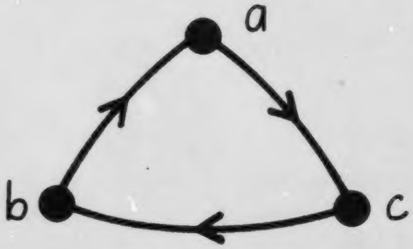


Figure 3

| | | | |
|---|---|---|---|
| c | | ● | |
| b | | | ● |
| a | ● | ● | ● |
| | a | b | c |

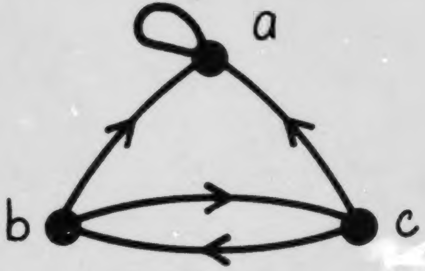


Figure 4

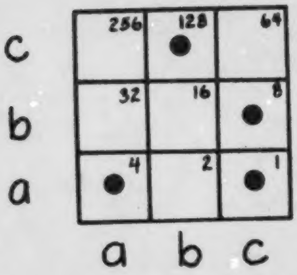
T: Nabu discovered that functions can be recognized easily with this code. Do you know Nabu's method?

Invite the class to give their opinions.

S: For a function, there is at most one checker in each column.

T: Very good. Nabu's other code is a binary code. He labels the squares of a three-by-three grid "1", "2", "4", "8", and so on.

Label the squares of the grid picture for Figure 1 as shown below.


c
b
a
256
128
64
32
16
8
4
2
1
a b c

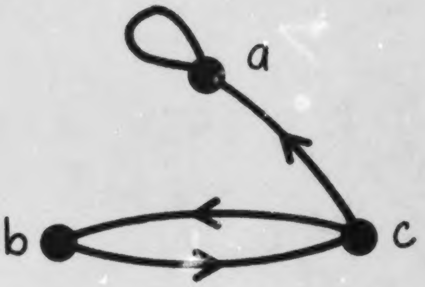


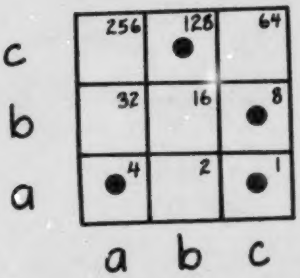
Figure 1

T: In the binary code, exactly one number is assigned to each grid picture (or arrow picture). Which number do we assign to Figure 1?

S: 141, because 1 + 4 + 8 + 128 = 141.

Indicate that the code number for Figure 1 is 141.

Code Number 141


c
b
a
256
128
64
32
16
8
4
2
1
a b c

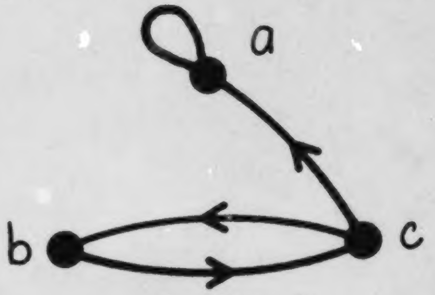


Figure 1

Ask the class to find the code number for Figure 2. Allow a few minutes for individual work while you observe and help students having difficulty. Call on a student who has the correct code number (68) to explain the coding at the board.

Code Number 68

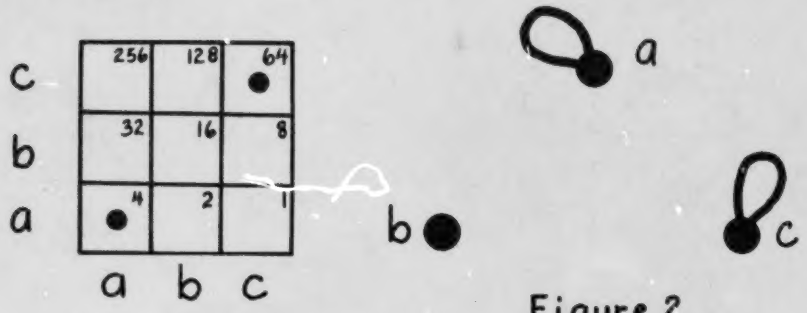

c
b
a
256
128
64
32
16
8
4
2
1
a
b
c
a
b
c

Figure 2

S: Label the squares of the grid like a binary abacus. There is one checker on the 4-square and one checker on the 64-square. So the code number is 4 + 64 or 68.

Ask the class to find the code number for Figure 3 and for Figure 4.

Code Number 266

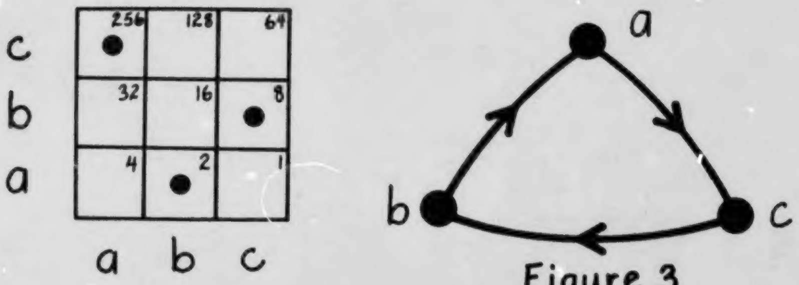

c
b
a
256
128
64
32
16
8
4
2
1
a
b
c
a
b
c

Figure 3

Code Number 143

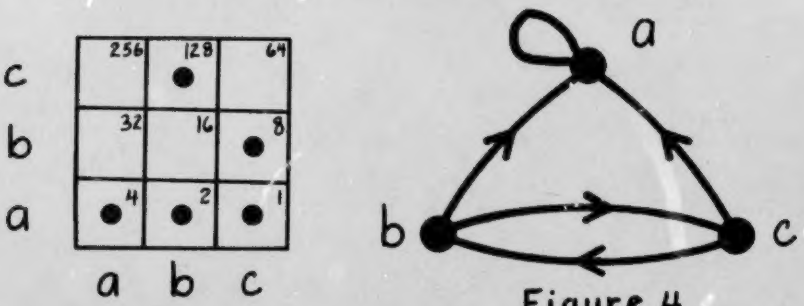

c
b
a
256
128
64
32
16
8
4
2
1
a
b
c
a
b
c

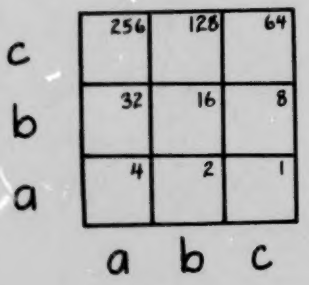
Figure 4

T: 100 is the code-number of a relation. Can you draw its arrow picture?

S: First draw a grid and label the squares "1", "2", "4", and so on. Then put checkers on the appropriate squares.

Draw and label the squares of a grid on the board.

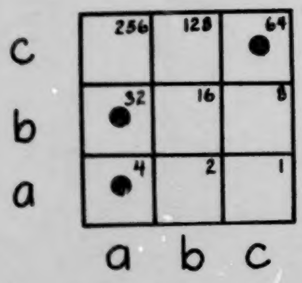
Code Number 100


c
b
a
256
128
64
32
16
8
4
2
1
a
b
c

T: Where do we put the checkers for 100?

S: Put one checker on the 4-square, one checker on the 32-square, and one checker on the 64-square.

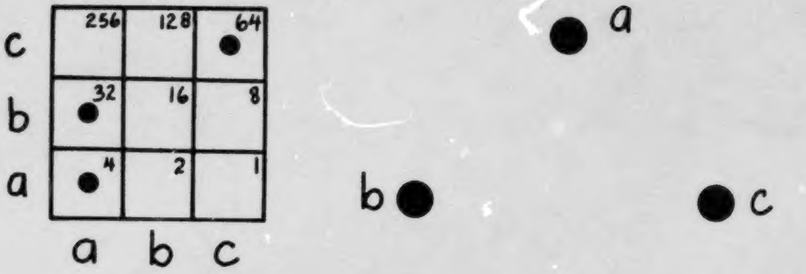
Code Number 100


c
b
a
256
128
64
32
16
8
4
2
1
a
b
c

# N 5

**S:** Next, draw three dots labeled "a", "b", and "c" for the arrow picture.

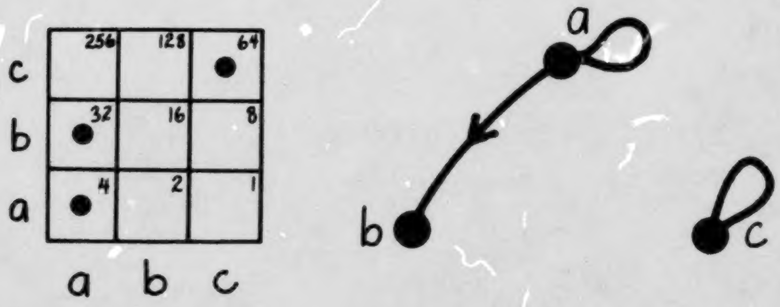
Code Number 100


c
b
a
256
128
64
32
16
8
4
2
1
a
b
c
a
b
c

Invite a student to draw the corresponding arrow picture and to give an explanation.

**S:** Look at the grid. There are two checkers in the first column. That means there are two arrows starting at a — one arrow ends at a (it's a loop) and one arrow ends at b. Furthermore, there is one checker in the third column so there is one arrow starting at c and ending at c — a loop.

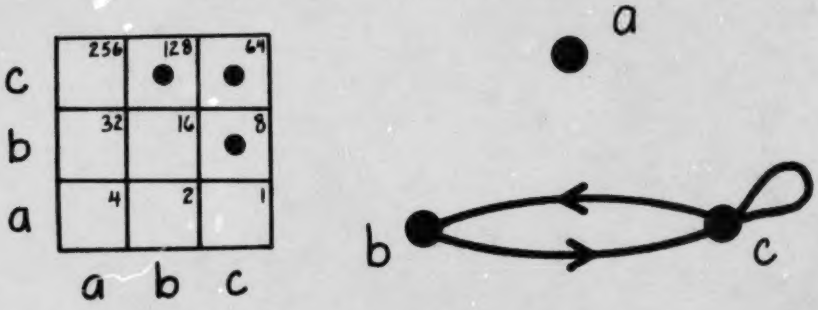
Code Number 100


c
b
a
256
128
64
32
16
8
4
2
1
a
b
c
a
b
c

**T:** Very good. Now, find the arrow picture having code number 200.

S: 200 is the double of 100, so move the checker on the 4-square to the 8-square and move the checker on the 32-square to the 64-square and move the checker on the 64-square to the 128-square.

Code Number 200

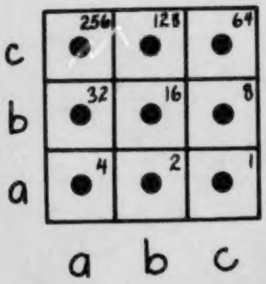


T: How many different relations do you think there are on a set of three objects?

Compare the students' guesses now with those made at the beginning of the lesson. Erase any guess the class believes is unreasonable (such as 3 or 4) and write the new guesses on the board.

T: We know each relation has exactly one code number and each code number is for exactly one relation. What is the largest possible code number for a relation on a set of three objects?

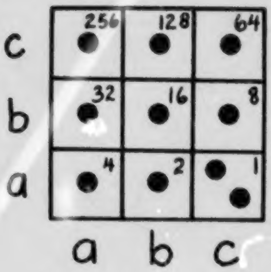
S: The largest possible code number is the one we get by placing a checker on each square of the grid.



# N 5

T: Which number is this? Can you find this number without doing much addition?

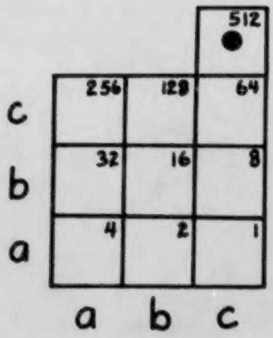
If no student suggests putting an extra checker on the 1-square, do so yourself.



T: Which number is this?

S: The number is 512. Exchange the two checkers on the 1-square for one checker on the 2-square. Then we have two checkers on the 2-square which can be exchanged for one checker on the 4-square. Keep doing this until there are two checkers on the 256-square which (on the binary abacus) is the same as one checker on the 512-square.

If you prefer, invite a student to make the exchanges.



204

**S:** The largest code-number is 511, 512 - 1 = 511.

If necessary, remind the class that an extra checker was put on the grid to get 512, so that a checker on every square is for 511.

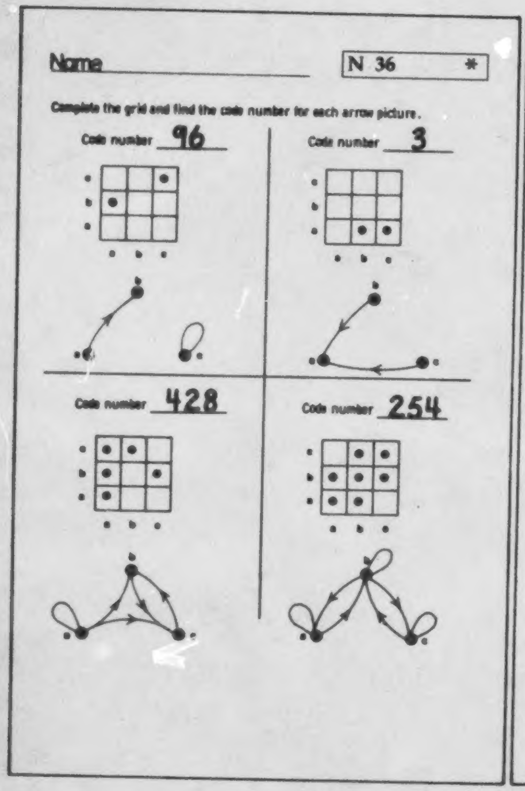
**T:** What is the smallest code-number?

**S:** 0; the relation with no arrows.

**T:** How many relations are there on a set of three objects?

**S:** Exactly one relation for each code number, 0 to 511; 512 relations.

Worksheets N36 and N37 are available for independent work.



Name ____________________ N 36 *

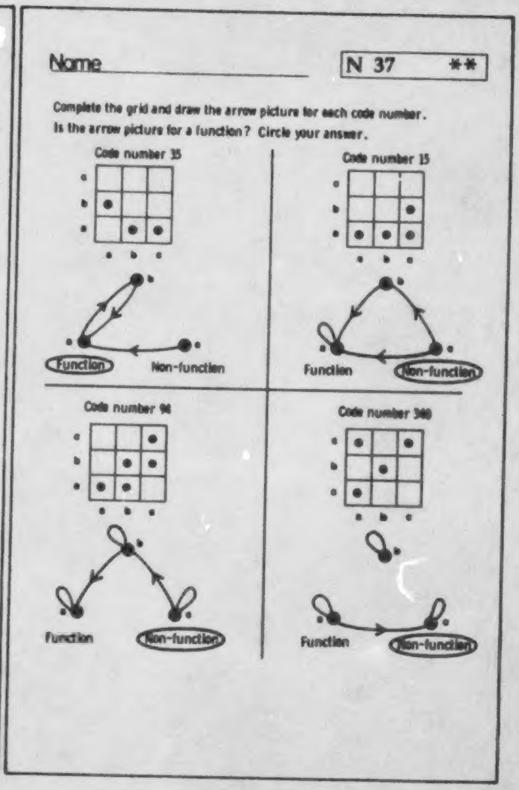
Complete the grid and find the code number for each arrow picture.

Code number 96

Code number 3

Code number 428

Code number 254



Name ____________________ N 37 **

Complete the grid and draw the arrow picture for each code number.
Is the arrow picture for a function? Circle your answer.

Code number 35

[✓ Function] Non-function

Code number 15

Function [✓ Non-function]

Code number 94

Function [✓ Non-function]

Code number 340

Function [✓ Non-function]

## 4.2 Counting Relations #2

CAPSULE LESSON SUMMARY

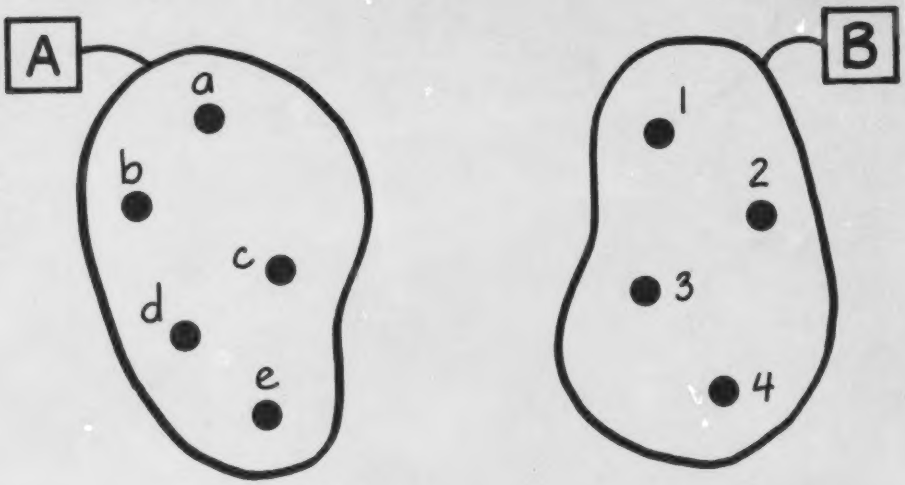
Count the number of different relations with arrows starting from dots in a set A (five members) and ending at dots in a set B (four members) by using codes, in particular a binary code, and assigning code numbers to the relations.

MATERIALS

Teacher: Colored chalk; magnetic checkers (optional)
Student: Paper; colored chalk

DESCRIPTION OF LESSON

Draw this string picture on the board.



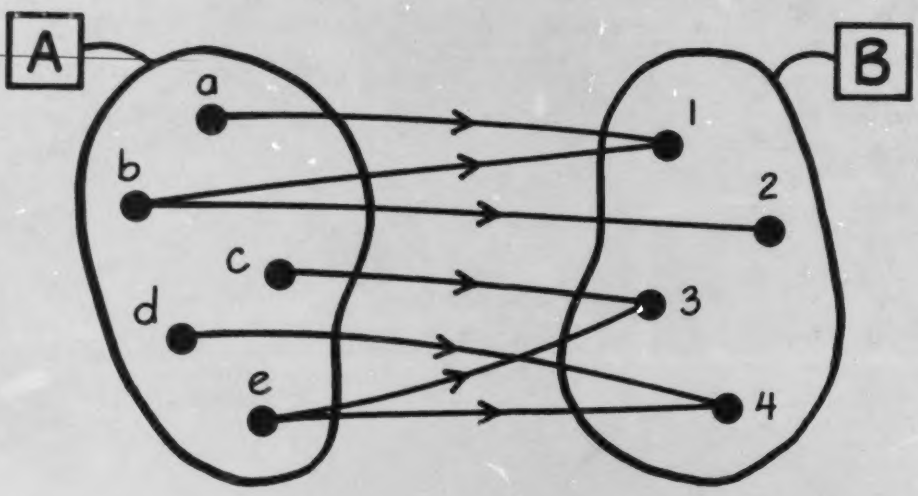
T: Copy these strings on a sheet of paper and draw any arrow picture you wish with arrows starting in A and ending in B.

If necessary, indicate that between a pair of dots (the first in A and the second in B) there is at most one arrow. Between two dots there can be one arrow or none at all, but two or more arrows would be redundant.

Allow several minutes for individual work, and observe the students' progress, giving help where necessary. Encourage faster workers to draw more than one arrow picture.

# N 20

After a while, call on a student to draw an arrow picture on the board. Suppose this arrow picture is drawn (of course, you will need to adapt the following discussion to fit the arrow picture on the board).



T: How many arrows are in this picture?

S: Seven.

T: What is the smallest number of arrows that could be drawn?

S: Zero.

T: Yes, and what is the largest number of arrows that could be drawn?

S: Twenty.

T: Why twenty?

S: We can draw arrows starting from a and ending at 1, 2, 3, and 4, respectively. This can be done also for b, c, d, and e. So there could be four arrows starting at each of the five dots in A and 5 x 4 = 20.

T: Look at the arrow picture on the board.
Is this picture for a function?

S: No, b and e are guilty points because two arrows start from each of them.

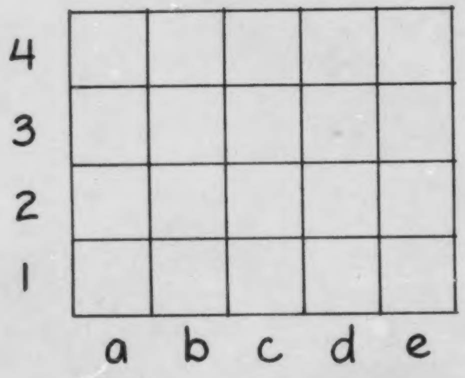
T: How many different arrow pictures with arrows from A to B could we draw?

Write the students' guesses on the board and save them for comparison to the exact number of such arrow pictures. The rest of this lesson will be directed at finding that number.

T: How could we calculate this number?

Some students might recall that in Lesson N4.1 a grid and the binary code were used to solve a similar problem. If this is not suggested, lead a discussion in this direction.

Draw this grid on the board.



## N 20

**T:** Previously, we used a grid to help us solve a similar problem. Here, the letters in A are listed below the columns; the numbers in B are listed to the left of the rows. Who can draw a grid picture for the arrow picture?

Invite a student to draw the picture and to explain the procedure.

**S:** There is an arrow starting at e and ending at 3. On the grid, put a checker (or draw a dot*) in the square where the column for e meets the row for 3. Likewise for each of the remaining six arrows.

With assistance from several students, complete the grid picture.

| | | | | | |
|---|---|---|---|---|---|
| 4 | | | | ● | ● |
| 3 | | | ● | | ● |
| 2 | | ● | | | |
| 1 | ● | ● | | | |
| | a | b | c | d | e |

**T:** Can we tell by looking at the grid picture that this relation is not a function?

**S:** There are two columns that have more than one checker in them. Those columns are for guilty points.

**T:** Yes. Now that we have the grid picture for our relation, what should we do next?

**S:** Find a code number for the relation. Label each square of the grid — "1", "2", "4", "8", and so on and use the binary code.

---

* If you do not have a magnetic board on which to put the grid, simply draw red dots for checkers.

On a separate grid, collectively label the squares.

| | | | | | |
|---|---|---|---|---|---|
| 4 | 524,288 | 262,144 | 131,072 | 65,536 | 32,768 |
| 3 | 16,384 | 8,192 | 4,096 | 2,048 | 1,024 |
| 2 | 512 | 256 | 128 | 64 | 32 |
| 1 | 16 | 8 | 4 | 2 | 1 |
| | a | b | c | d | e |

T: How do we calculate the code number of our arrow picture?

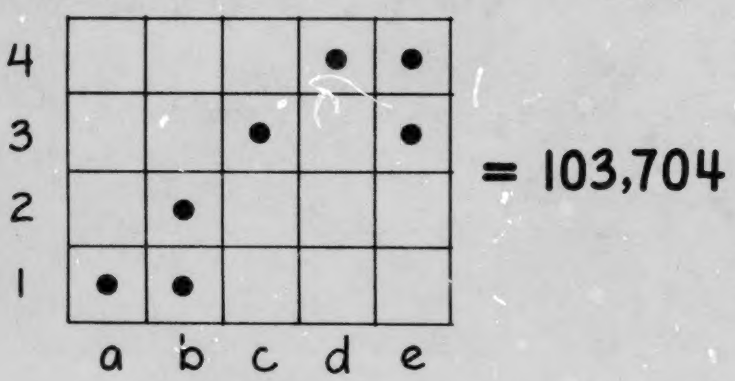
S: Add the numbers of the squares where there are checkers.

Write the addition problem on the board and ask the class to do the calculation.

$$\begin{array}{lr} & 65{,}536 \\ & 32{,}768 \\ & 4{,}096 \\ & 1{,}024 \\ & 256 \\ & 16 \\ + & 8 \\ \hline & 103{,}704 \end{array}$$

# N 20

**S:** The code number is 103,704.

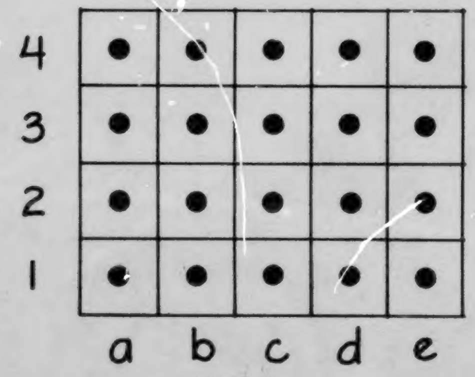


**T:** 103,704 is one code number. What is the smallest code number?

**S:** 0; it is the code number of the relation with no arrows.

**T:** What is the largest possible code number?

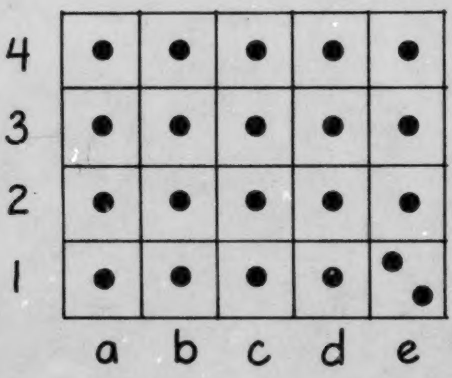
**S:** Put a checker on each square of the grid — that shows the largest number.



**T:** Which number is this? How can we calculate this number easily?

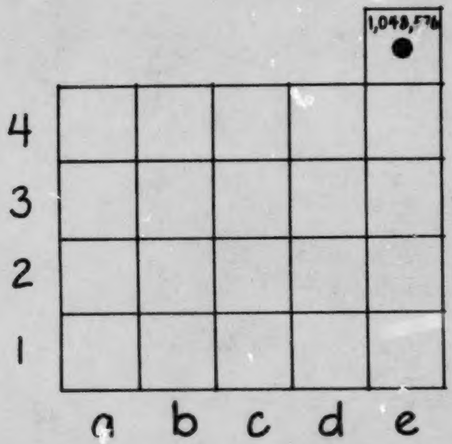
**S:** Put another checker on the 1-square. Then the number is 1 more than the largest code number.

If no student suggests this strategy, do so yourself.



T: Which number is this?

S: The number is 2 × 524,288 or 1,048,576. I thought about making plays. Two checkers on the 1-square is the same as one checker on the 2-square. Then two checkers on the 2-square is the same as one checker on the 4-square, and so on until there are two checkers on the 524,288-square. Then two checkers there is the same as one checker on the 1,048,576-square or 2 × 524,288.



If you prefer, invite a student to make the plays with checkers.

S: The largest code number is 1,048,575; 1,048,576 - 1 = 1,048,575.

# N 20

**T:** Do you think there is one code number for each arrow picture?

**S:** Yes. You can always draw a grid picture and find a code-number.

**T:** Do you think we can start from any code-number between 0 and 1,048,575 and draw an arrow picture? For example, let's try to draw an arrow picture for 5,000.

Let the students work on this problem independently before solving it collectively.

**T:** First let's make the grid picture for 5,000.

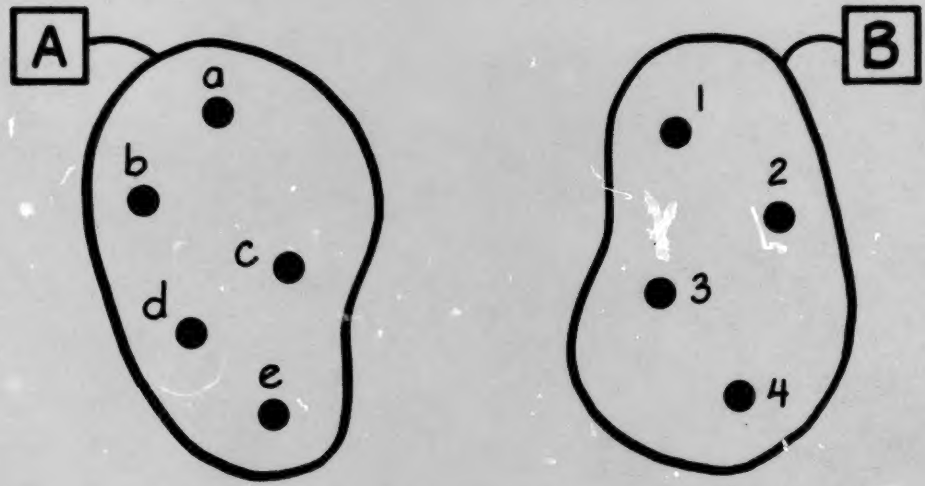
Draw a grid on the board and ask students to place checkers for 5,000.

**S:** Place a checker on the 4,096-square, one on the 512-square, one on the 256-square, one on the 128-square, and one on the 8-square.

5,000 =

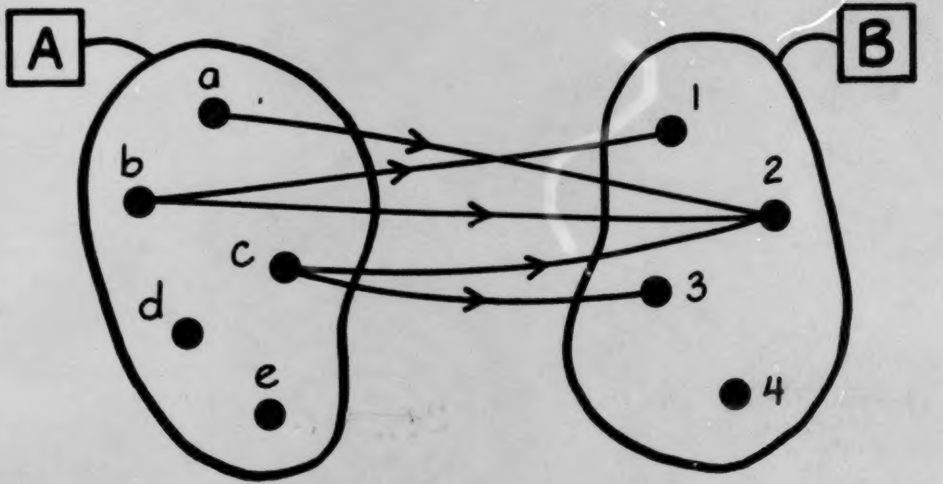
| | a | b | c | d | e |
|---|---|---|---|---|---|
| 4 | 524,288 | 262,144 | 131,072 | 65,536 | 32,768 |
| 3 | 16,384 | 8,192 | 4,096 ● | 2,048 | 1,024 |
| 2 | 512 ● | 256 ● | 128 ● | 64 | 32 |
| 1 | 16 | 8 ● | 4 | 2 | 1 |

Draw the two strings, A and B, on the board.



Invite a student to draw the corresponding arrow picture and to give an explanation.

S: There is one checker in the first column. That checker is for an arrow that starts at a and ends at 2. The checkers in the second column are for arrows starting at b, one ends at 1 and the other ends at 2. There are two arrows starting at c because there are two checkers in the third column, one ends at 2 and the other ends at 3.



# N 20

T: Do you think there is an arrow picture for each code-number between 0 and 1,048,575?

S: Yes. We can always use the grid picture to draw the arrow picture.

T: Good. For each arrow picture there is exactly one code number and for each code number there is exactly one arrow picture.

T: How many different relations (or arrow pictures) with arrows from A to B are there?

S: One relation for each code number, 0 to 1,048,575; 1,048,576 relations in all.

Be sure to compare the students' estimates to 1,048,576.

## 4.3 How Many Functions?

CAPSULE LESSON SUMMARY

Review the notion of a function and count the number of different functions with arrows starting from dots in a set A (five members) and ending at dots in a set B (four members). Compare the number of functions to the number of relations (found in Lesson N4.2). Repeat the counting activity for both relations and functions with arrows from a new set A to a new set B.

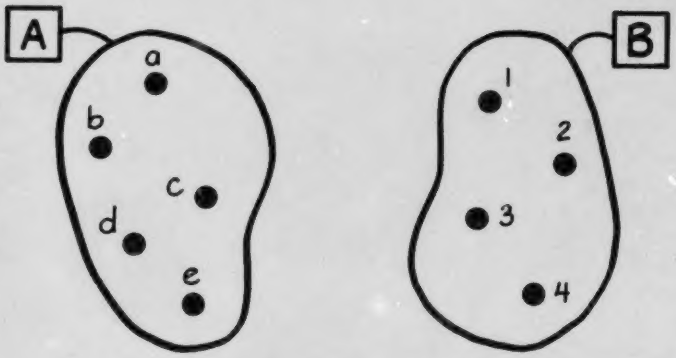
MATERIALS

Teacher: Colored chalk; magnetic checkers (optional)
Student: Paper; colored chalk

### DESCRIPTION OF LESSON

#### Exercise 1

Draw this string picture on the board.



T: Do you remember the lesson where we counted the number of relations with arrows from A to B?

Let the class recall as much as they can about that lesson. Bring out in the discussion that:

- A grid and the binary code were used to help with the counting.
- There are more than one million (1,048,576) different relations with arrows from A to B.

# N 22

T: We know there are more than one million different relations with arrows from A to B. Some of these are functions. What do we mean when we say "function"?

S: There are no guilty points.

S: Starting at each dot in A there can be at most one arrow.

Point out that this means guilty points have two or more arrows starting at them; it is okay to have no arrow or exactly one arrow starting at a dot.

Ask the class to draw on their paper an example of a function with arrows from A to B. Invite a few students to draw their pictures on the board. For example,

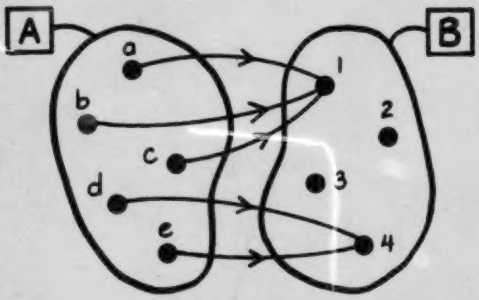


Figure 1

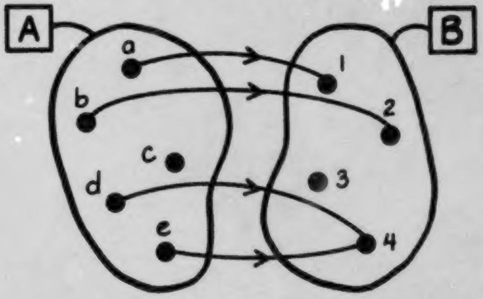


Figure 2

T: How many different functions with arrows from A to B are there?

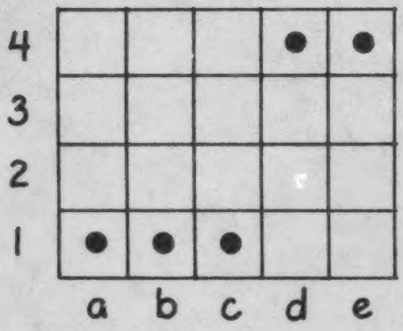
Write the students' estimates on the board for future reference.

T: How can we calculate this number?

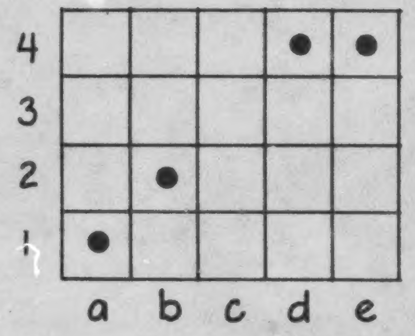
S: We could use a grid.

T: That's a good idea. On your paper draw a grid picture for the function you just drew.

Ask two students to draw the grid pictures for the functions on the board.



Grid for Figure 1



Grid for Figure 2

T: Look carefully at the grid pictures. How can we tell by looking at the grid picture of a relation that it is a function?

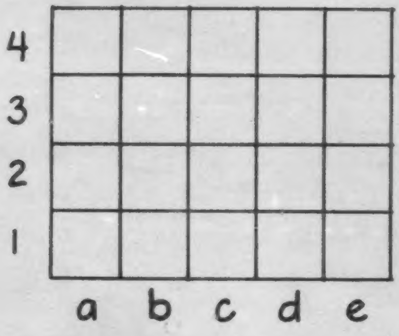
Let the students consider the question for a couple minutes.

S: There is at most one checker in each column.

T: Yes. How can we use this fact to help with the counting? In other words, how many ways can we put at most one checker in each column of the grid?

Give the class an opportunity to discuss this situation. Some students may suggest that there is a multiplication calculation to do to count the number of functions with arrows from A to B. If this idea is not suggested, make it yourself. Then solve the problem collectively.

Draw this grid on the board.



# N 22

Cover all but column a of the grid with a piece of paper.

T: In how many ways can we put at most one checker in column a?

S: Five ways; no checker or one checker in four ways.

Uncover column b.

T: In how many ways can we put at most one checker in column a and at most one checker in column b?

S: Twenty-five ways. There are five ways to put at most one checker in column a and, with each of those ways, there are five ways to put at most one checker in column b; so there are 5 x 5 ways.

Uncover column c.

T: In how many ways can we put at most one checker in each of columns a, b, and c?

S: 125; 5 x 5 x 5 = 125.

Continue in this manner until all five columns are uncovered. The students should conclude that there are 5 x 5 x 5 x 5 x 5 = 3,125 functions with arrows from A to B.

Compare the students' estimates to 3,125.

T: There are about three thousand functions and about one million relations with arrows from A to B. Imagine that we have each possible arrow picture on a slip of paper. Then we put the slips of paper into a bag and randomly select one slip of paper from the bag. What is the probability that we get a function?

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S: About 0.003 (3,000 ÷ 1,000,000).

T: So there are about three chances out of one thousand of getting a function.

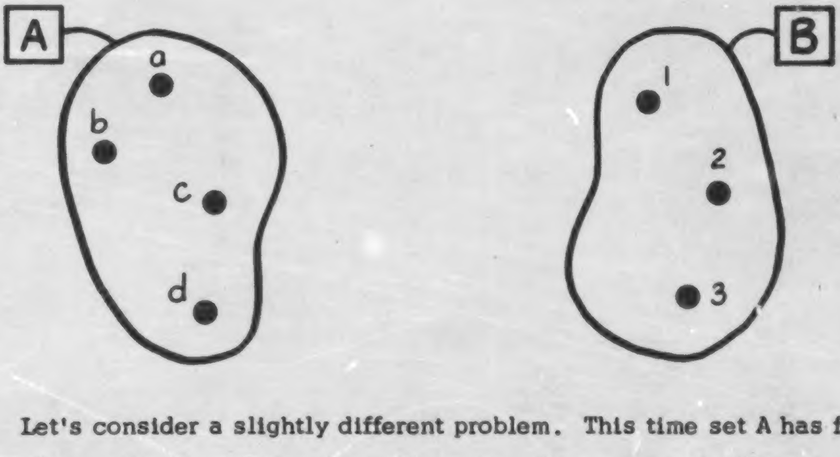
Write this information on the board.

$$\frac{3,000}{1,000,000} = 0.003$$

Exercise 2

NOTE: The purpose of this exercise is to provide a similar, but easier, problem that students can solve independently. If your class is quite capable, you may want to give less direction and more time for independent work. A weak class will demand more direction on your part.

Draw this string picture on the board.



T: Let's consider a slightly different problem. This time set A has four elements and set B has three elements. On your paper, draw an arrow picture of a relation with arrows from A to B.

# N 22

Let the students work independently. Ask two or three students to draw their pictures on the board. For example,

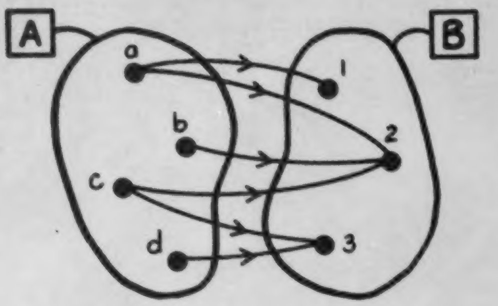


Figure 1

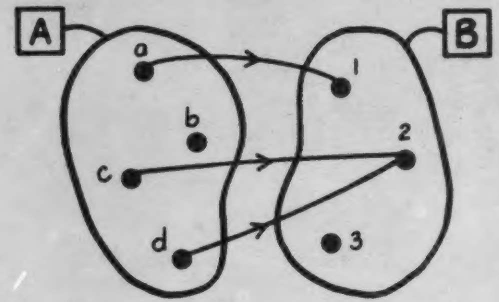


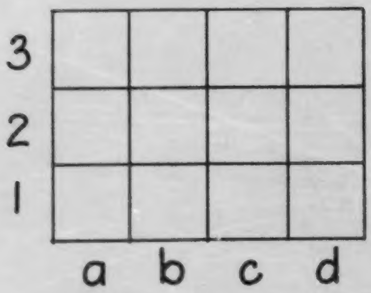
Figure 2

T: How many different relations with arrows from A to B do you think there are? How could we count them?

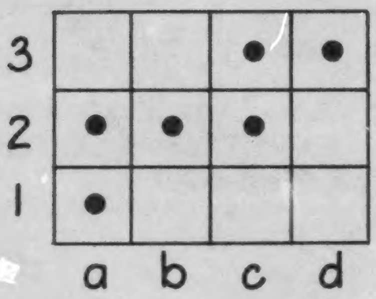
Encourage students to estimate the number of relations and discuss their methods of counting.

S: We could use a grid and the binary code.

If this suggestion is not made by the class, make it yourself. Then draw this grid on the board.



Instruct the students to draw the grid pictures for the relations they just drew. While they are working, invite a student to put checkers in the grid for one of the relations on the board.



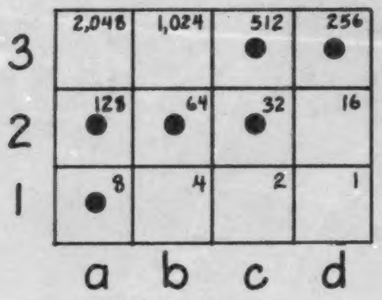
Grid for Figure 1

**T:** What should we do next?

**S:** Find the code number. Label the squares of the grid "1", "2", "4", "8", and so on.

Label the squares of the grid and give the students a few minutes to find the code number.

Code Number 1,000


2,048
1,024
512
256
128
64
32
16
8
4
2
1
3
2
1
a
b
c
d

Grid for Figure 1

**T:** What is the smallest code number?

**S:** 0.

**T:** ... the largest code number?

**S:** One checker on every square; the code number is 4,096 - 1 = 4,095.

# N 22

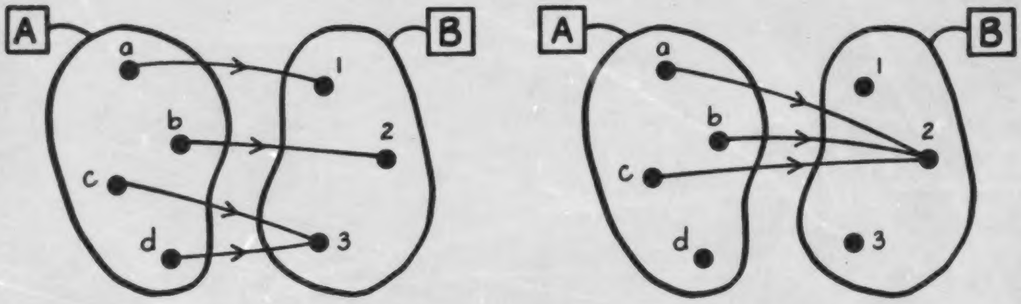
If necessary, put checkers on the grid and make exchanges to illustrate that the largest code number is 4,095.

T: Good. So 0 is the smallest code number and 4,095 is the largest code number. How many relations with arrows from A to B are there?

S: There are 4,096 code numbers from 0 to 4,095, so there are 4,096 relations with arrows from A to B.

T: Yes, there are exactly 4,096 different relations. Now let's think about functions. On your paper, draw a function with arrows from A to B.

Allow several minutes for independent work. Walk around and observe the students' work, giving assistance when needed. Invite two or three students to draw their pictures on the board.

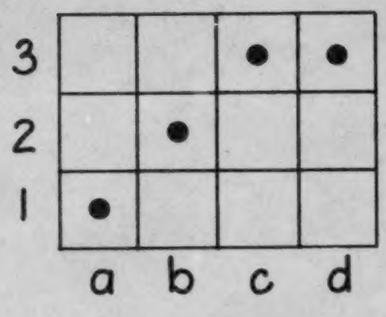


T: We have seen some examples of functions with arrows from A to B. How many different functions are there? How can we count them?

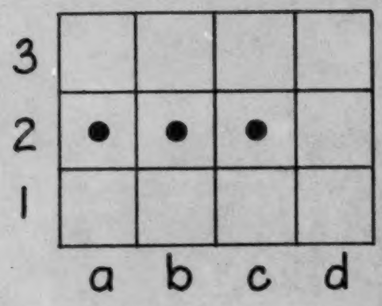
S: We can use the grid for counting.

T: On your paper, draw a grid picture for the function you just drew.

Call on students to draw grid pictures for the functions on the board.



Grid for Figure 1



Grid for Figure 2

**T:** What do you notice about the grid picture for a function?

**S:** There is at most one checker in each column.

**T:** Good. How can we use this observation to count the number of functions with arrows from A to B?

**S:** See how many ways we can put at most one checker in each column of the grid.

If this method is not suggested by someone, give it yourself.

**T:** Very good. How many ways can at most one checker be placed in each column of the grid?

Let the students work independently a few minutes. After a while, solve this problem collectively in the manner of Exercise 1. The students should conclude that there are 4 x 4 x 4 x 4 = 256 ways to put at most one checker in each column of the grid and thus 256 functions with arrows from A to B.

**T:** Imagine we have a picture of every relation with arrows from A to B on pieces of paper in a bag. What is the probability of getting a function if we select at random a piece of paper?

**S:** 256 chances out of 4,096.

**T:** Yes. To make the calculation easier let's say about 250 chances out of 4,000.

Write this information on the board.

$$\frac{250}{4,000}$$

**T:** What is another name for $\frac{250}{4,000}$?

**S:** "$\frac{25}{400}$" or "$\frac{1}{16}$".

$$\frac{250}{4,000} = \frac{25}{40} = \frac{1}{16}$$

**T:** So there is about one chance out of sixteen of getting a function.

## 4.4 How Many Permutations?

CAPSULE LESSON SUMMARY

Play a game with six students resulting in a permutation (special kind of function with exactly one arrow starting and ending at each dot) on a set of six objects. Observe that the arrow picture for permutations may have from one to six pieces. Count the number of permutations and then the number that have connected (one-piece) arrow pictures. Find that when playing the game, the probability of getting a connected arrow picture is 1/6.

MATERIALS

| | |
|---|---|
| Teacher: | Small pieces of paper; colored chalk; magnetic checkers (optional) |
| Student: | Paper; colored pencils; Worksheets N38 and N39 |

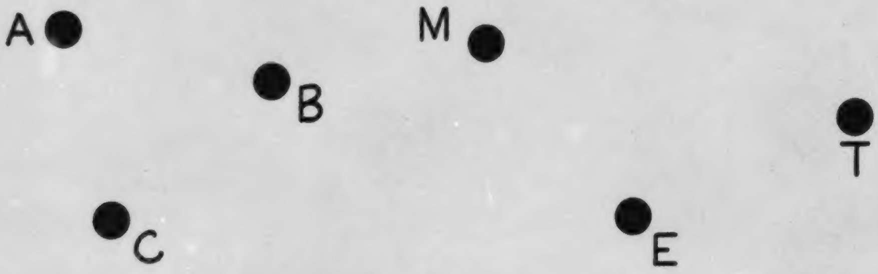
### DESCRIPTION OF LESSON

In preparation for this lesson, choose six students in your class whose first names start with different letters. Write each name on a separate piece of note-size paper and fold it so that the name is not visible.

To make the description of the lesson easier to follow, we use the names of six students in one of our development classes: Angela, Barbara, Charles, Edward, Mark, and Troyce.

Exercise 1

Ask the six students to stand at the front of the class. Draw a dot for each of them on the board and label the dots with their first initials.



# N 35

T: I have six pieces of paper. Each piece of paper has one of your names written on it. I will mix up the pieces of paper and give one to each of you. Do not open it yet. When I say "Go!", unfold the paper and read the name. Then with your right hand take the left hand of the person whose name is on your piece of paper.

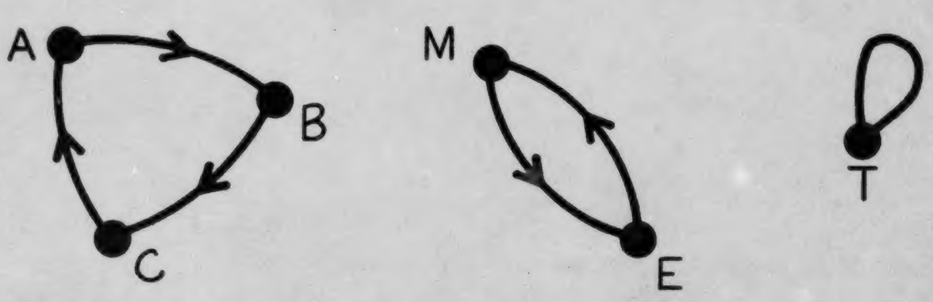
Pass out the papers and say "Go!" Watch the students to make sure they follow instructions.

T: Let's draw some red arrows for "I hold your left hand with my right hand." Who can draw a red arrow in our picture on the board?

S: Angela is holding Barbara's left hand. We can draw an arrow from A to B.

S: Troyce received her own name -- she holds her left hand with her own right hand, so we can draw a loop at T.

Invite students not participating in the game to draw arrows and complete the arrow picture. Your arrow picture might look like this.



NOTE: In the picture, exactly one arrow must start at each dot and exactly one arrow must end at each dot.

Ask the six students to take their seats.

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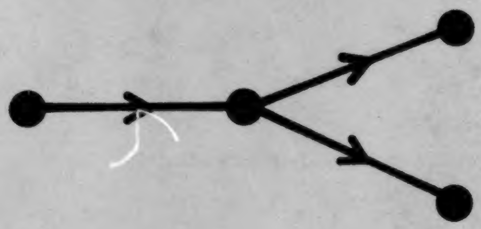
T: On your paper, draw six well-spaced dots. Label them with the letters we used on the board.

Imagine that we play the game a second time. Draw an arrow picture for this imaginary game. Your picture should not be exactly the same as the one on the board. If we play the game again, it probably will have a different result from the first game.

Allow several minutes for the students to work on their arrow pictures. Monitor students' progress and give help to those who have difficulty getting started.

A few students may have difficulty drawing an appropriate arrow picture. For example, students may have this as part of their arrow picture.



T: What do you think of this situation?

S: That's impossible. No one has three hands.

S: The arrow picture shows someone got two names. That can't happen.

T: Look at the arrow picture on the board. Is it a function?

S: Yes, at each dot at most one arrow starts.

T: Good. In fact, at each dot exactly one arrow starts. What do you notice about the number of arrows that end at each dot?

S: There is exactly one arrow that ends at each dot.

T: So, there is exactly one arrow starting at each dot and there is exactly one arrow ending at each dot.

How many pieces does the picture on the board have?

S: Three.

T: How many (disconnected) pieces do your pictures have?

S: Four pieces.

S: Six pieces.

T: How could you have an arrow picture with six pieces?

S: Six loops.

T: What is the smallest number of pieces we could have?

S: One piece. The six students could hold hands in a circle.

T: What is the largest number of pieces?

S: Six; six loops.

T: How many different arrow pictures could we draw for six people playing this game?

Record the students' estimates in a corner of the board and save them for future reference.

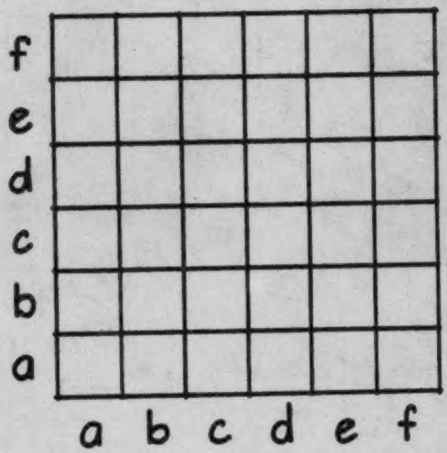
T: How do you suggest we solve this problem?

Give the class a few minutes to study the problem.

S: We could use a grid.

If this suggestion is not forthcoming, make it yourself.

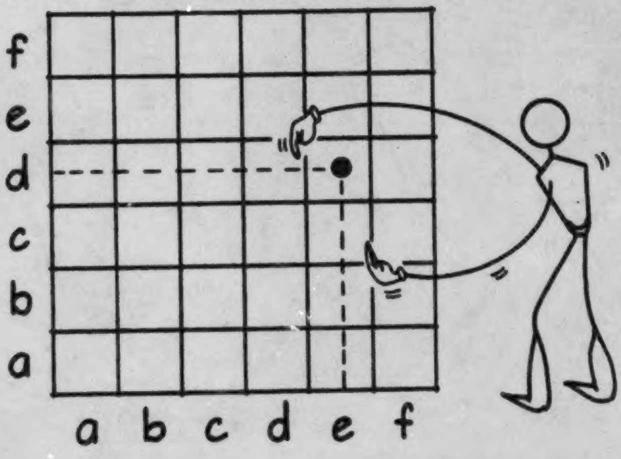
Distribute Worksheet N38 and draw this grid on the board.



T: Suppose students a, b, c, d, e, and f are playing this game. Complete the grid for the arrow picture on the worksheet.

Let the students work independently. When most students have finished, invite someone to give an explanation at the board.

S: There is an arrow starting at e and ending at d. Put a checker in the square where the column for e (along the bottom) meets the row for d (along the side).



# N 35

The completed grid is given in the following illustration.

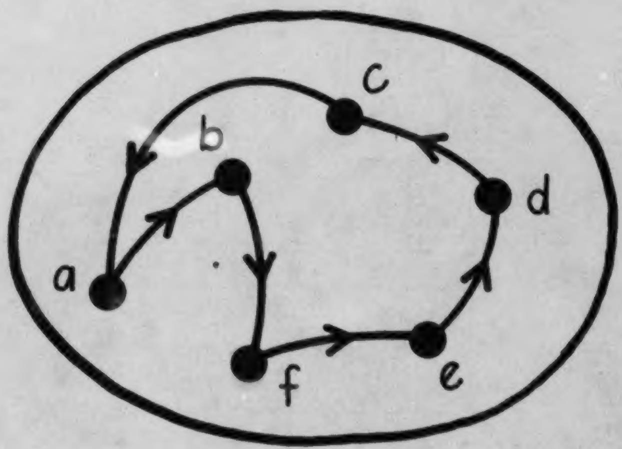
| | | | | | | |
|---|---|---|---|---|---|---|
| f | ● | | | | | |
| e | | | | ● | | |
| d | | | | | ● | |
| c | | ● | | | | |
| b | | | | | | ● |
| a | | | ● | | | |
| | a | b | c | d | e | f |

T: Do you notice anything interesting about the placement of checkers in the grid ?

S: There is exactly one checker in each row and in each column.

T: Yes, because exactly one arrow starts at each dot and exactly one arrow ends at each dot.

Distribute Worksheet N39 and ask the students to draw the corresponding arrow picture. After a few minutes, invite students to draw the arrow picture as shown below on the board.



T: How does the grid help us to count the number of different arrow pictures?

Give the class an opportunity to discuss this question.

S: We can count the ways six checkers can be put in the grid so that there is exactly one checker in each row and in each column.

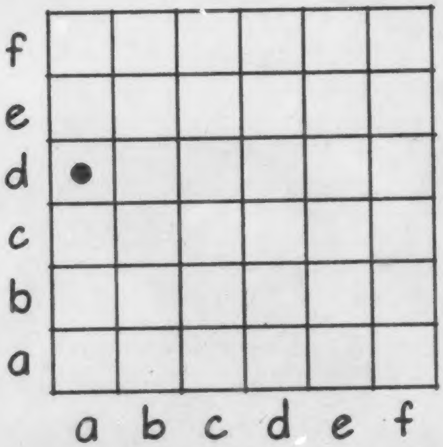
If this suggestion is not made by your class, make it yourself. Lead the class to see this situation as a multiplication problem. Cover all but column a with a piece of paper.

T: First, we put a checker in column a. How many choices do we have?

S: Six choices.

T: Yes, suppose we place the checker here.

Place a checker anywhere in column a you wish.



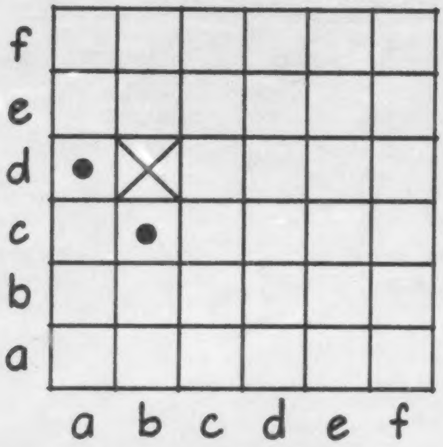
Uncover column b.

T: Now, we put a checker in column b. How many choices do we have?

S: Only five choices. We cannot put a checker here (points to the square for (b, d)). If we did, we would have a row with two checkers in it.

T: Very good. Each column and each row must have exactly one checker in it.

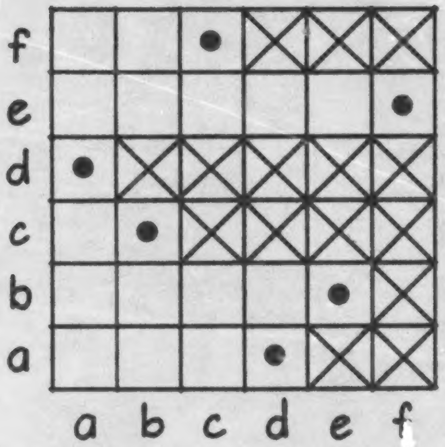
Place a checker in any one of the five squares (a forbidden square is indicated with "×").



T: How many ways can we place two checkers in the first two columns so that exactly one checker is in each row and in each column?

S: 6 × 5 = 30; 30 ways to place the first two checkers.

Continue in this manner analyzing each column. The students should conclude that there are 6 × 5 × 4 × 3 × 2 × 1 = 720 different arrow pictures. Your grid may look like the one shown in the next illustration at the conclusion of your analysis.



Be sure to compare the students' estimates to 720.

Exercise 2

T: We have noticed that sometimes the arrow picture is in several pieces. In fact, we said that there could be as few as one piece or as many as six pieces.

When we play this game, randomly giving out the pieces of paper, what is the probability of getting a one-piece or a connected arrow picture?

Record the students' estimates on the board. Save the estimates for later comparison.

T: How can we calculate this probability?

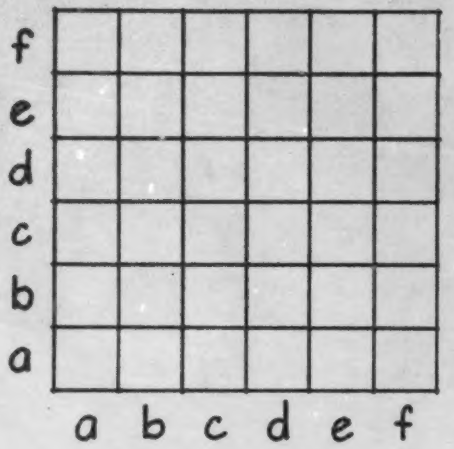
Give the students an opportunity to express their opinions.

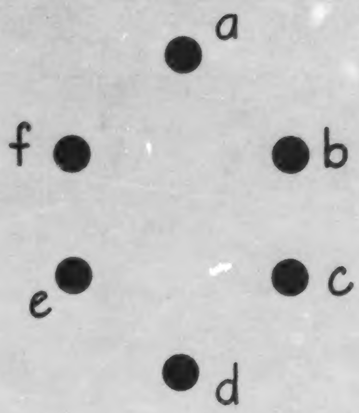
S: We should calculate how many different one-piece arrow pictures there are among the 720 arrow pictures.

T: Very good. Let's look at the grid and try to put six checkers in the grid in such a way that the corresponding arrow picture is connected.

# N 35

Clear the grid and draw six labeled dots on the board.



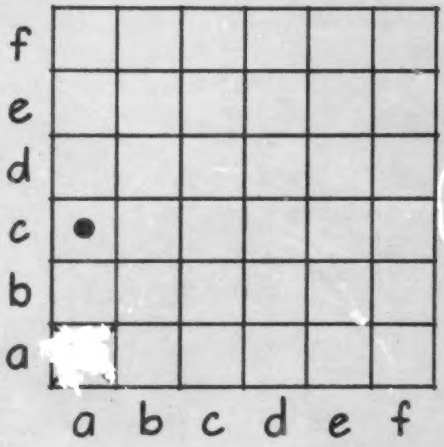


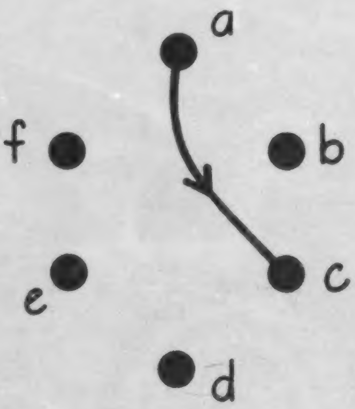
T (holding up a checker): Where could we place the first checker in column a?

S: Anywhere except in the first row. We cannot place the checker in row a because we would have a loop and eventually more than one piece.

S: So there are only five possibilities.

Cross out the forbidden square and place a checker in any one of the five remaining squares. Also draw the corresponding arrow.





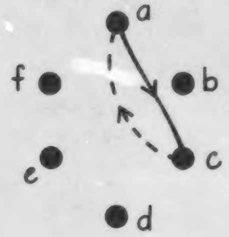
T (holding up a second checker): Now, where could we place the second checker in column c? Why do I ask about column c?

S: The first arrow ends at c so, to keep the connection, we should look next for the arrow starting at c.

T: Good. Can we place this checker anywhere in column c?

S: No. We cannot put it it row c because then there would be two checkers in that row.

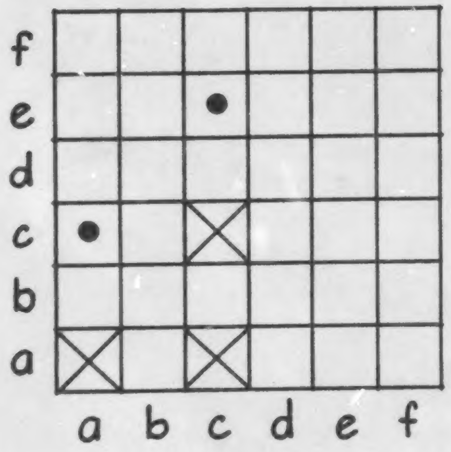
S: Also, we cannot put it in row a, because then we would have an arrow starting at c and ending at a and so eventually more than one piece.

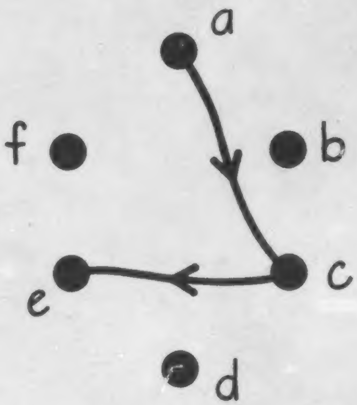


S: There are only four ways to place the checker.

Cross out the two forbidden squares and place a checker in one of the four remaining squares.

Draw the corresponding arrow.





# N 35

**T:** How many ways may we place the first two checkers in this grid?

**S:** Twenty ways, 5 x 4 = 20.

**T** (holding up a checker): In which column do you suggest we put the third checker?

**S:** The second arrow ends at e, so put the third checker in column e.

**T:** Yes, can we put the checker anywhere in column e?

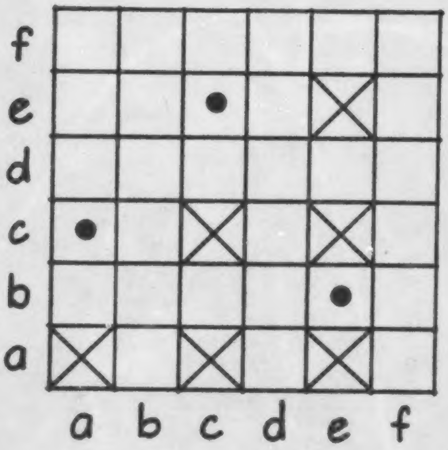
**S:** Not in row a — that would make a cycle and eventually give us more than one piece.

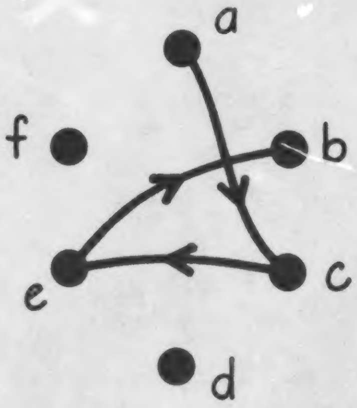
**S:** Not in row c, because that would give us two checkers in row c.

**S:** Not in row e — that would give two checkers in row e.

**S:** There are just three ways to place the third checker.

Cross out the three forbidden squares, place a checker in one of the three remaining squares, and draw the corresponding arrow.



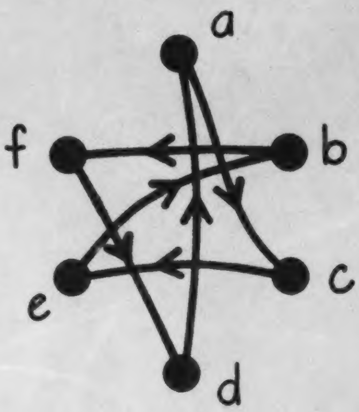


**T:** How many ways may we place the first three checkers in this grid?

S: Sixty ways, 5 x 4 x 3 = 60.

Continue the analysis in this manner for the three remaining columns. The students should conclude that there are two possibilities for column b and one possibility each for column d and column f. So there are 120 (5 x 4 x 3 x 2 x 1 x 1) different ways to have a connected arrow picture. Your grid might look like the one shown below.

| | | | | | | |
|---|---|---|---|---|---|---|
| f | | ● | | ☒ | | ☒ |
| e | | ☒ | ● | ☒ | ☒ | ☒ |
| d | | | | ☒ | | ● |
| c | ● | ☒ | ☒ | ☒ | ☒ | ☒ |
| b | | ☒ | | ☒ | ● | ☒ |
| a | ☒ | ☒ | ☒ | ● | ☒ | ☒ |
| | a | b | c | d | e | f |



T: What is the probability of getting a connected (one-piece) arrow picture?

S: 120 chances out of 720.

Write this information on the board.

$$\frac{120}{720}$$

T: What is a simpler fraction for $\frac{120}{720}$?

## N 35

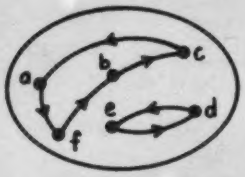
**S:** "$\frac{1}{6}$".

$$\frac{120}{720} = \frac{1}{6}$$

**T:** So there is about one chance out of six of getting a connected arrow picture.

Name ____________________ N 38

Fill in the grid for this arrow picture.



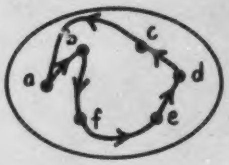
| | | | | | | |
|---|---|---|---|---|---|---|
| f | ● | | | | | |
| e | | | | ● | | |
| d | | | | | ● | |
| c | | ● | | | | |
| b | | | | | | ● |
| a | | | ● | | | |
| | a | b | c | d | e | f |

---

Name ____________________ N 39

Draw the arrow picture for this grid.

| | | | | | | |
|---|---|---|---|---|---|---|
| f | | ● | | | | |
| e | | | | | | ● |
| d | | | | | ● | |
| c | | | | ● | | |
| b | ● | | | | | |
| a | | | ● | | | |
| | a | b | c | d | e | f |



# 5

# MULTIPLES AND DIVISORS

This chapter extends the experiences with multiples and divisors of numbers in situations which lead to tests for divisibility and which introduce new relations and operations such as $=_4$ and exponentiation ($a * b = a^b$). Other situations review the prime factor relation, the least common multiple relation, and the greatest common divisor relation.

Detective stories, arrow diagrams, and string pictures provide opportunities to explore both new and familiar ideas and to discover some properties of multiples, divisors, and the various relations and operations involving them. A probability problem suggests that certain divisibility tests be used in its solution.

CAPSULE LESSON SUMMARIES

| | |
|---|---|
| Section 5.1: | Introduce the relation $=_4$ by playing a Guess My Rule game with arrows. Observe that there are four pieces to the arrow picture and that in one piece all the numbers are multiples of 4, or all the numbers are 1 more than multiples of 4, or all the numbers are 2 more than multiples of 4, or all the numbers are 3 more than multiples of 4. Complete a truth table for the relation $=_4$ using the fact that two numbers are equal modulo 4 if and only if they belong in the same string (piece of the arrow picture).<br>(Worksheets N40, N41, N42, N43) |
| Section 5.2: | Review the greatest common divisor relation $\sqcap$ and the least common multiple relation $\sqcup$ using string pictures. Present a detective story where the clues involve $\sqcup$, $\sqcap$, and a string picture where the strings are for positive divisors of certain numbers.<br>(Worksheets N44, N45, N46) |

| | |
|---|---|
| Section 5.3: | Explore a new rule for a relation where the arrows mean that the ending number is the sum of the digits of the starting number. Use the arrow picture for this relation to begin looking at a test for divisibility by 9.<br>(Worksheets N47, N48) |
| Section 5.4: | Fill in the square with the smallest possible whole number for number stories such as "356 = △ + □" where "△" must be for some multiple of 9. Relate this activity to the arrow picture for 0's rule in Section 5.3, and find a test for deciding whether or not a number is a multiple of 9 or of 3. (A number is a multiple of 9 (3) if the sum of its digits is a multiple of 9 (3).) Provide missing digits so that numbers are multiples of 9. |
| Section 5.5: | Introduce exponential notation using a Guess My Rule game for an operation $*$ where $a * b = a^b$. Write various numbers as products of prime numbers making use of the exponential notation. Solve a detective story where the clues involve prime divisors of the secret number, the relation $=_4$, and the hand-calculator relation $\boxed{-}\ \boxed{6}\ \boxed{=}$ . . . . |
| Section 5.6: | Review the prime factor relation and find the prime factor distance between several pairs of numbers. Solve a detective story where the clues involve prime factor distance, the relation $=_3$, and the hand-calculator relation $\boxed{+}\ \boxed{5}\ \boxed{=}$ . . . . |

Section 5.7: Examine characteristics of the sum of two multiples of 4, the sum of a multiple of 4 and a non-multiple of 4, and the sum of two non-multiples of 4. Solve a detective story in which the clues involve multiples of 4, patterns generated by moving one checker in a configuration on the Minicomputer, numbers which can be put on the Minicomputer with exactly one weighted checker, the relation $=_3$, and the hand-calculator relation $\boxed{-}$ $\boxed{7}$ $\boxed{=}$ . . . .
(Worksheets N49, N50, N51)

Section 5.8: Find all the positive divisors from 2 to 10 of some selected numbers. Review tests for deciding whether or not a given number is divisible by 2, by 3, by 4, by 5, by 6, by 8, by 9, or by 10. Consider the numbers which can be formed by randomly arranging four digits and find the probabilities of getting numbers divisible by 2, by 3, by 4, by 8, by 9, by 5, and by 6.
(Worksheets N52, N53, N54)

# N 7

## 5.1 The Relation $=_4$

CAPSULE LESSON SUMMARY

Introduce the relation $=_4$ by playing a Guess My Rule game with arrows. Observe that there are four pieces to the arrow picture and that in one piece all the numbers are multiples of 4, or all the numbers are 1 more than multiples of 4, or all the numbers are 2 more than multiples of 4, or all the numbers are 3 more than multiples of 4. Complete a truth table for the relation $=_4$ using the fact that two numbers are equal modulo 4 if and only if they belong in the same string (piece of the arrow picture).

MATERIALS

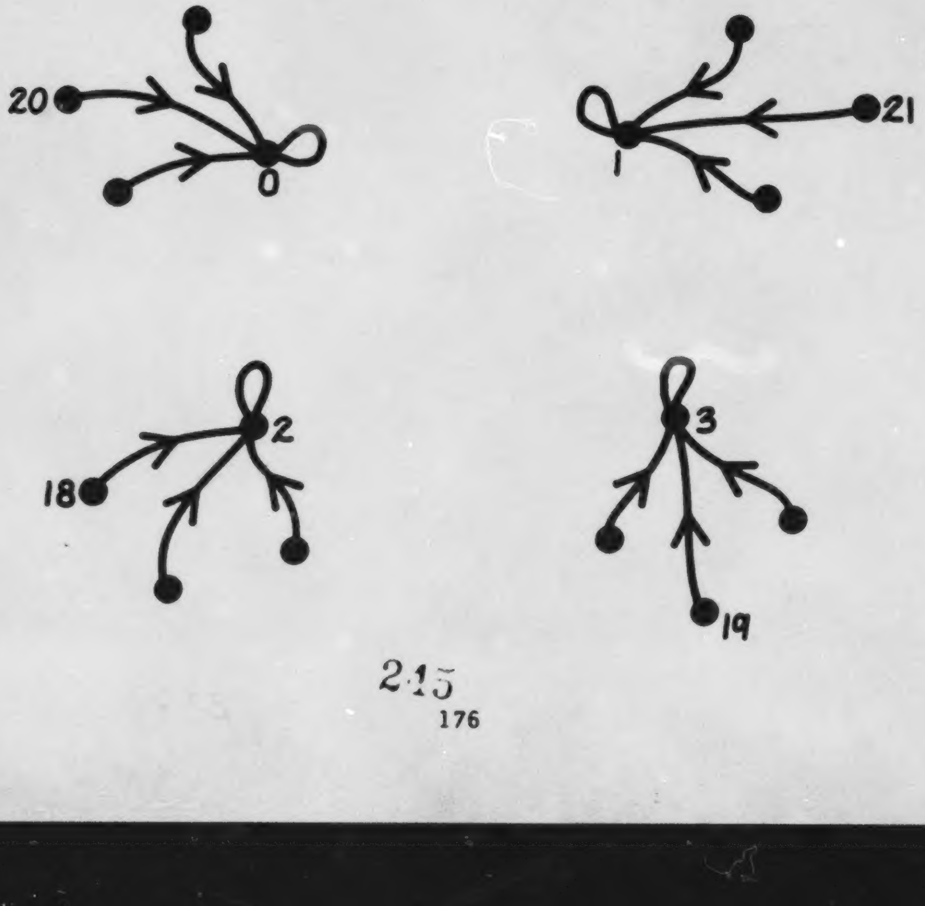
Teacher: Colored chalk

Student: Paper; pencil; Worksheets N40, N41, N42, and N43

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.

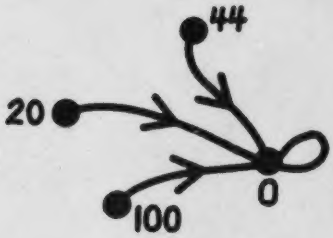


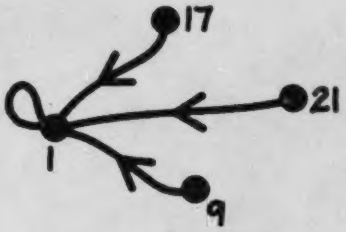
T: I have a secret rule for drawing red arrows. Can you guess my rule? There are several clues in this picture. Study the picture and try to find labels for other dots. I'll tell you if you are correct.

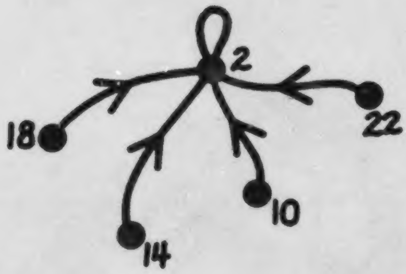
NOTE: The rule for the red arrows is: the starting number of an arrow ending at 0 is a multiple of 4, of an arrow ending at 1 is 1 more than a multiple of 4, of an arrow ending at 2 is 2 more than a multiple of 4, and of an arrow ending at 3 is 3 more than a multiple of 4. Do not announce the rule at this time.

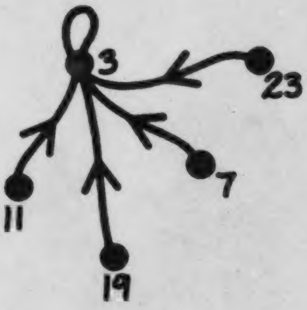
Allow several minutes for the class to study the clues. Then, one at a time, invite students to label dots and each time announce whether or not a dot is labeled correctly (erase incorrect labels).

Continue this activity until it is obvious that many students know the rule. Add more arrows to your picture if necessary. We cannot predict which numbers your students will choose, but your picture might look like the one in the next illustration.









# N 7

T: Who can explain my rule?

S: Each arrow that ends at 0 starts at a multiple of 4. Each arrow that ends at 1 starts at a number which is 1 more than a multiple of 4. Each arrow that ends at 2 starts at a number which is 2 more than a multiple of 4. Each arrow that ends at 3 starts a number which is 3 more than a multiple 4.

Erase all the labels for the dots except "0", "1", "2", and "3". Write this list of numbers on the board.

37
137
537
2,337
987,637

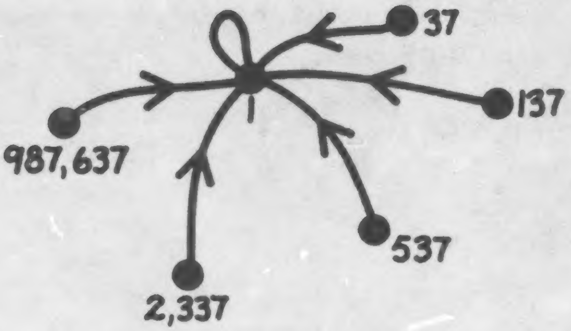
T: Where do these numbers belong in the arrow picture?

S: 37 is the starting number of an arrow ending at 1, because $(4 \times 9) + 1 = 37$ so 37 is 1 more than a multiple of 4.

S: Also, 137 is the starting number of an arrow ending at 1. 100 is a multiple of 4 and 36 is a multiple of 4, so 136 is a multiple of 4. 137 is 1 more than a multiple of 4.

The class should conclude that each number listed on the board is 1 more than a multiple of 4.

S: Each of those numbers is the starting number of an arrow ending at 1. A multiple of 100 is also a multiple of 4 and each number on the board is a multiple of 100 plus 36 plus 1, so each is 1 more than a multiple of 4.



Distribute Worksheet N40 and write this list of numbers on the board.

50
28
38
35
41

51
75
502
499
997

76
132
134
301
138

179
1,005
376

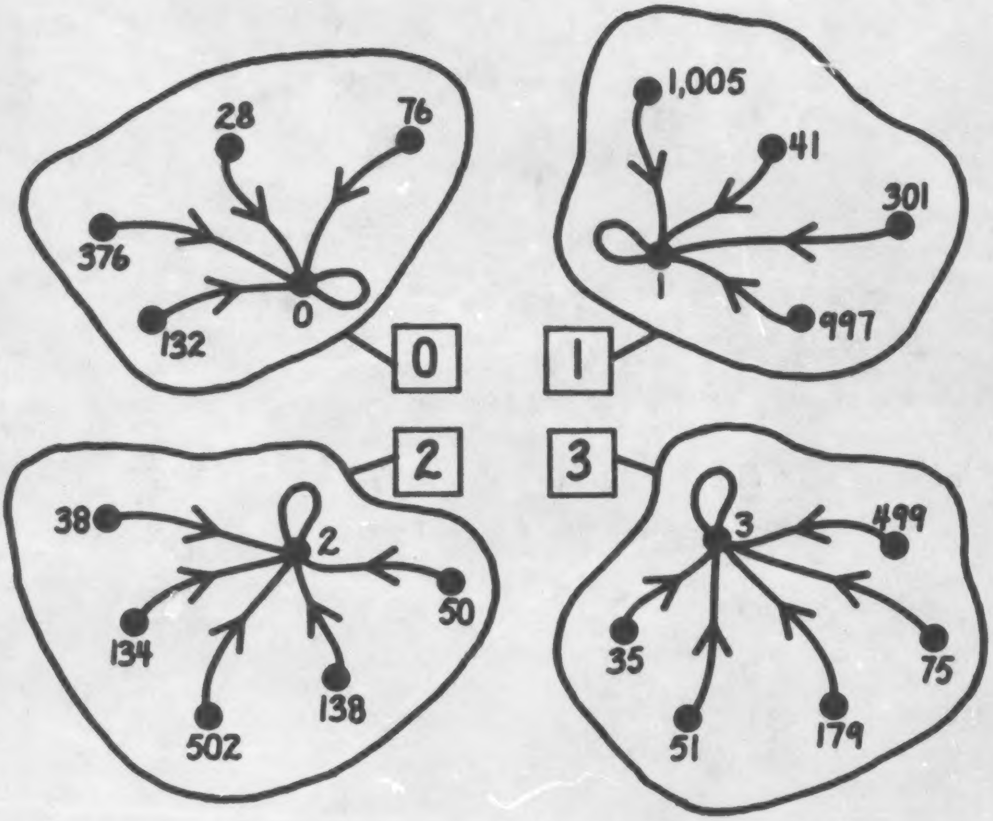
NOTE: Feel free to adjust the numbers to the numerical level of your class.

T: Put these numbers in the arrow picture on your worksheet.

After some time for independent work, call on students to put the numbers in the picture on the board. See page 186 for a completed picture.

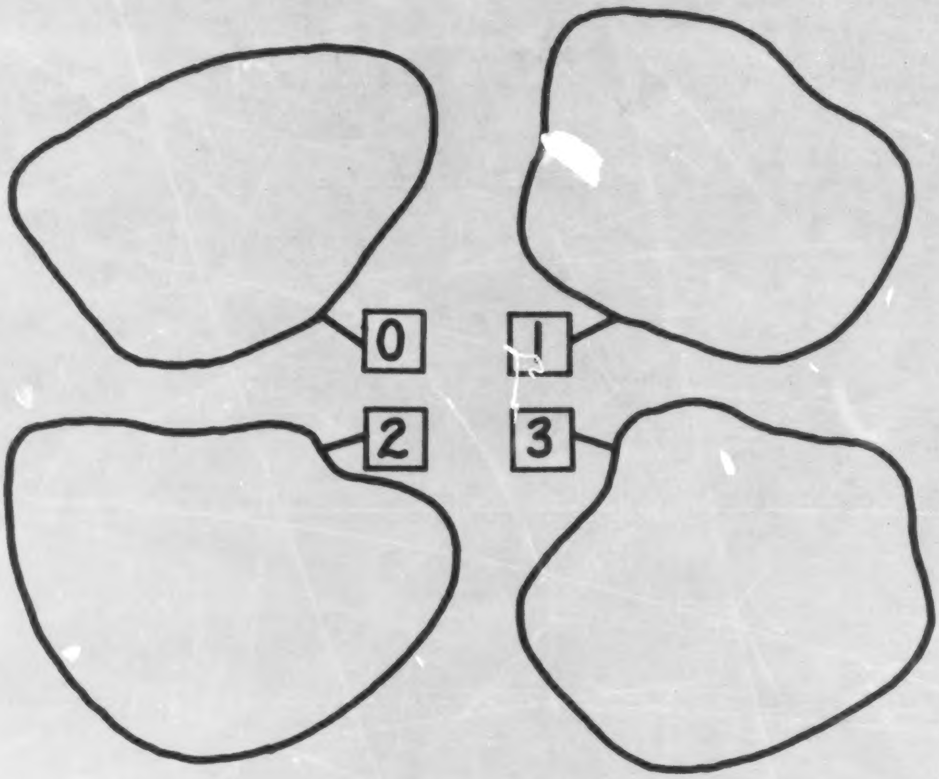
Exercise 2

Draw a string around each piece of the arrow picture. (See the next illustration.)


28
76
376
132
0
1,005
41
301
1
997
0
1
2
3
38
2
50
134
502
138
499
3
35
75
51
179

T (pointing to the appropriate strings): The 0-string is for multiples of 4, the 1-string is for numbers which are 1 more than multiples of 4, the 2-string is for numbers which are 2 more than multiples of 4, and the 3-string is for numbers which are 3 more than multiples of 4.

Erase the contents of the four strings and write this number story on the board.



$$\text{Pif} =_4 \text{Paf}$$

T (pointing to "$=_4$"): Pif is equal modulo 4 to Paf if and only if Pif and Paf are in the same string. Who can show where we could put dots for Pif and Paf?

Invite a student to show where to draw dots for Pif and Paf. Accept any answer as long as both dots are in the same string.

# N 7

Draw a table on the board under the number story.

| Pif $=_4$ | Paf |
|---|---|
| | |
| | |
| | |
| | |

T: Copy this table and fill in numbers that Pif and Paf could be. Remember, Pif is equal modulo 4 to Paf.

Allow a few minutes for independent work while you help those students who need it. When many students are finished, collectively fill in the table on the board. For example,

| Pif $=_4$ | Paf |
|---|---|
| 24 | 64 |
| 37 | 49 |
| 107 | 407 |
| 15 | 19 |
| 60 | 84 |

T: Look closely at the table. Do you notice anything interesting?

Accept several observations leading to the following one.

S: 64 - 24 = 40 and 40 is a multiple of 4. The difference is a multiple of 4 for each pair of numbers.

T: Yes. Another way to think of the relation $=_4$ is: two numbers are equal modulo 4 if and only if their difference is a multiple of 4.

Distribute Worksheet N41 and draw this table on the board.

| $=_4$ | 10 | 21 | 52 | 79 | 98 |
|---|---|---|---|---|---|
| 8 | | | | | |
| 75 | | | | | |
| 11 | | | | | |
| 1,002 | | | | | |
| 25 | | | | | |
| 78 | | | | | |

T: If two numbers are equal modulo 4, put a "T" in the corresponding box. If two numbers are not equal modulo 4, put an "F". For example, $8 \neq_4 10$, so an "F" goes in this box (write "F" in the appropriate box).

| $=_4$ | 10 | 21 | 52 | 79 | 98 |
|---|---|---|---|---|---|
| 8 | F | | | | |
| 75 | | | | | |
| 11 | | | | | |
| 1,002 | | | | | |
| 25 | | | | | |
| 78 | | | | | |

Instruct the class to go on filling in the table. After a few minutes of individual work, check the table collectively by discussing a method for knowing whether a particular entry is "T" or "F".

# N 7

**T:** Here is one way to help us in filling in this table. (Point to the four strings.) In which string is 8? ... is 75?

**S:** 8 is a multiple of 4, so 8 is in the 0-string.

**S:** 75 is 3 more than a multiple of 4 (75 = 72 + 3), so 75 is in the 3-string.

Label your table as indicated below.

|  | $=_4$ | 10 | 21 | 52 | 79 | 98 |
|---|---|---|---|---|---|---|
| $\boxed{0}$ | 8 | F |  |  |  |  |
| $\boxed{3}$ | 75 |  |  |  |  |  |
|  | 11 |  |  |  |  |  |
|  | 1,002 |  |  |  |  |  |
|  | 25 |  |  |  |  |  |
|  | 78 |  |  |  |  |  |

Ask the class to determine in which string each of the remaining numbers belongs.

|  |  | $\boxed{2}$ | $\boxed{1}$ | $\boxed{0}$ | $\boxed{3}$ | $\boxed{2}$ |
|---|---|---|---|---|---|---|
|  | $=_4$ | 10 | 21 | 52 | 79 | 98 |
| $\boxed{0}$ | 8 | F |  |  |  |  |
| $\boxed{3}$ | 75 |  |  |  |  |  |
| $\boxed{3}$ | 11 |  |  |  |  |  |
| $\boxed{2}$ | 1,002 |  |  |  |  |  |
| $\boxed{1}$ | 25 |  |  |  |  |  |
| $\boxed{2}$ | 78 |  |  |  |  |  |

**T:** Where in the table could we write "T"?

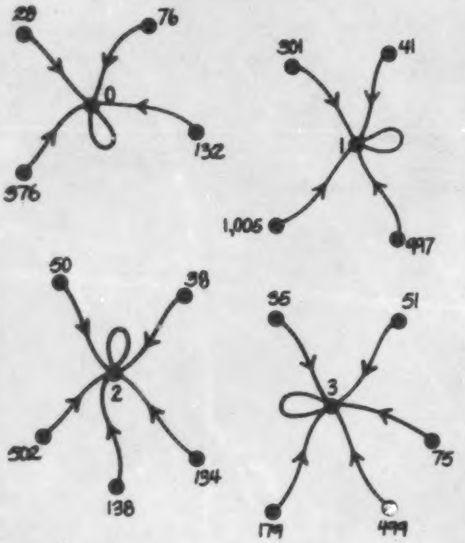
**S:** 75 and 79 each belong to the 3-string, so $75 =_4 79$.

| | | $\boxed{2}$ | $\boxed{1}$ | $\boxed{0}$ | $\boxed{3}$ | $\boxed{2}$ |
|---|---|---|---|---|---|---|
| | $=_4$ | 10 | 21 | 52 | 79 | 98 |
| $\boxed{0}$ | 8 | F | | | | |
| $\boxed{3}$ | 75 | | | | T | |
| $\boxed{3}$ | 11 | | | | | |
| $\boxed{2}$ | 1,002 | | | | | |
| $\boxed{1}$ | 25 | | | | | |
| $\boxed{2}$ | 78 | | | | | |

The class should find that there are eight "T"'s in the table. See page 186 for a completed table.

Worksheets N42 and N43 are available.

Name ______________________ N 40


28
76
0
132
576
301
41
1
1,005
997
50
38
2
502
138
194
35
51
3
75
179
499

Name ______________________ N 41

Fill in the table.

| $=_4$ | 10 | 21 | 52 | 79 | 98 |
|---|---|---|---|---|---|
| 8 | F | F | T | F | F |
| 75 | F | F | F | T | F |
| 11 | F | F | F | T | F |
| 1,002 | T | F | F | F | T |
| 25 | F | T | F | F | F |
| 78 | T | F | F | F | T |

Name ______________________ N 42 *

Complete the table.

| $=_4$ | 81 | 73 | 28 | Bim | 70 | 77 |
|---|---|---|---|---|---|---|
| 55 | F | F | F | T | F | F |
| 92 | F | F | T | F | F | F |
| Bam | F | F | F | T | F | F |
| 194 | F | F | F | F | T | F |
| 201 | T | T | F | F | F | T |
| 216 | F | F | T | F | F | F |

Name ______________________ N 43 **

First clue

Flip is a secret whole number between 10 and 20.
Flap is a secret whole number between 40 and 50.

Second clue

| $=_4$ | Blop | 15 | 52 |
|---|---|---|---|
| Blip | True | False | False |
| 25 | False | F | F |
| 71 | False | T | F |

Complete this table and write the four couples of numbers that (Blip, Blop) could be.

(14 . 42) (14 . 46) (18 . 42) (18 . 46)

Third clue

Blip $=_8$ Blop

Blip is 18 . Blop is 42 .

## 5.2 The Relations $\sqcup$ and $\sqcap$

CAPSULE LESSON SUMMARY

Review the greatest common divisor relation $\sqcap$ and the least common multiple relation $\sqcup$ using string pictures. Present a detective story where the clues involve $\sqcup$, $\sqcap$, and a string picture where the strings are for positive divisors of certain numbers.

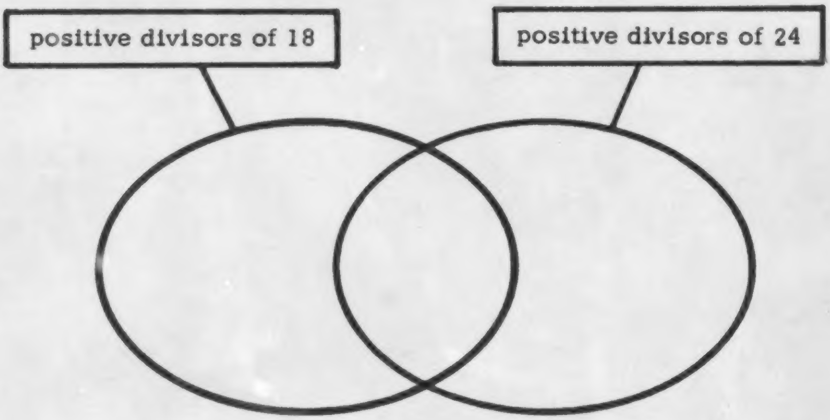
MATERIALS

| | |
|---|---|
| Teacher: | Colored chalk |
| Student: | Paper; colored pencils; Worksheets N44, N45, and and N46 |

DESCRIPTION OF LESSON

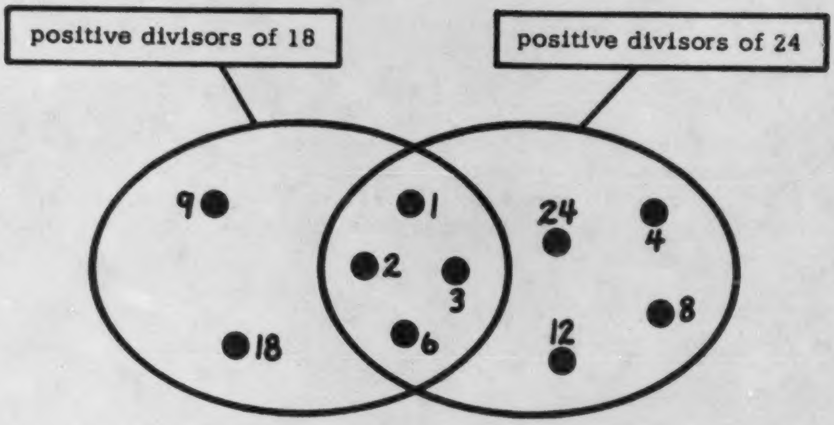
Exercise 1

Draw this string picture on the board.



T: Copy this string picture and put the positive divisors of 18 and of 24 in the picture.

Give the students a few minutes to work independently. When most students have finished, check the work collectively by inviting students to complete the picture at the board.


positive divisors of 18
positive divisors of 24
9
18
1
2
3
6
24
4
8
12

T: What are the common divisors of 18 and 24?

S: 1, 2, 3, and 6.

T: What is the greatest common divisor of 18 and 24?

S: 6.

Write this number story on the board.

$$18 \sqcap 24 = 6$$

T: The greatest common divisor of 18 and 24 is 6. Look at the common divisors of 18 and 24 again. What do you notice about the common divisors?

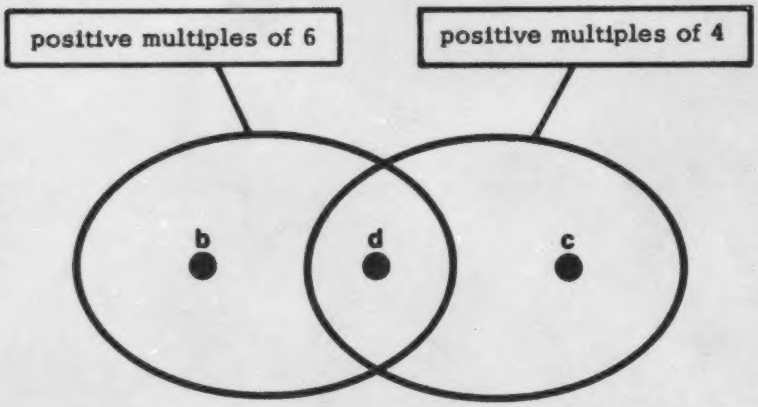
S: They are the positive divisors of 6 (the greatest common divisor).

Write the next set of problems on the board (answers are in the boxes) and ask the students to solve them. Feel free to adjust the problems to the numerical level of your class. Encourage students having difficulty to use string pictures as an aid. After a while check the work collectively.

$$12 \sqcap 15 = \boxed{3}$$
$$24 \sqcap 40 = \boxed{8}$$
$$4 \sqcap 9 = \boxed{1}$$
$$28 \sqcap 42 = \boxed{14}$$
$$75 \sqcap 135 = \boxed{15}$$

Exercise 2

Draw this string picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to b): Which numbers could be here?

S: 6, 18, 30, 42, 54, and so on.

S: Any positive multiple of 6 which is not a multiple of 4.

T (pointing to c): Which numbers could be here?

**S:** 4, 8, 16, 20, 28, 32, 40, and so on.

**S:** Any positive multiple of 4 which is not a multiple of 6.

**T (pointing to d):** Which numbers could be here?

**S:** 12, 24, 36, 48, and so on.

List the numbers on the board as they are suggested by the students.

12 24 36 48 60 72 ...

**T:** What do you notice about these numbers?

**S:** They are multiples of 12.

**S:** They are multiples of 6 and of 4.

**T:** Yes, these numbers are common multiples of 6 and 4. Which positive number is the least common multiple of 6 and 4?

**S:** 12.

Write this number story on the board.

$$6 \sqcup 4 = 12$$

NOTE: If you wish, point out that the symbol "$\sqcup$" is used for the least common positive multiple.

**T:** The least common multiple of 6 and 4 is 12. Look at the common multiples of 6 and 4 again. What do you notice about the common multiples?

S: Each common multiple is a multiple of 12 (the least common multiple).

Write these problems on the board (answers are in boxes) and ask the students to solve them. Feel free to adjust the problems to the numerical level of your class. Encourage students having difficulty to use string pictures as an aid. Check the work collectively.

$$10 \sqcup 15 = \boxed{30}$$
$$4 \sqcup 7 = \boxed{28}$$
$$12 \sqcup 10 = \boxed{60}$$
$$7 \sqcup 7 = \boxed{7}$$
$$3 \sqcup 6 = \boxed{6}$$

## Exercise 3

Present this detective story about a secret positive number named "Fa".

**First clue**

Write this information on the board.

$$10 \sqcup \text{Fa} = 30$$

T: Fa is a secret whole number. What information does this clue give us about Fa?

S: 30 is the least common multiple of 10 and Fa.

S: Fa is a positive divisor of 30.

T: Yes, write the positive divisors of 30 on your paper.

List the positive divisors of 30 on the board as the students give them to you.

$$1, 2, 3, 5, 6, 10, 15, 30$$

T: Which of these numbers cannot be Fa?

S: Fa cannot be 1 because $10 \sqcup 1 = 10$, not 30.

S: Fa cannot be 2, 5, or 10. The least common multiple of 10 and each of these numbers is 10, not 30.

Cross off from the list the numbers which Fa cannot be.

T: Do you notice anything interesting about the numbers we crossed off?

S: Each number we crossed off is a positive divisor of 10.

The students should conclude that Fa is 3, 6, 15, or 30.

### Second clue

Write this information on the board.

$$24 \sqcap Fa = 3$$

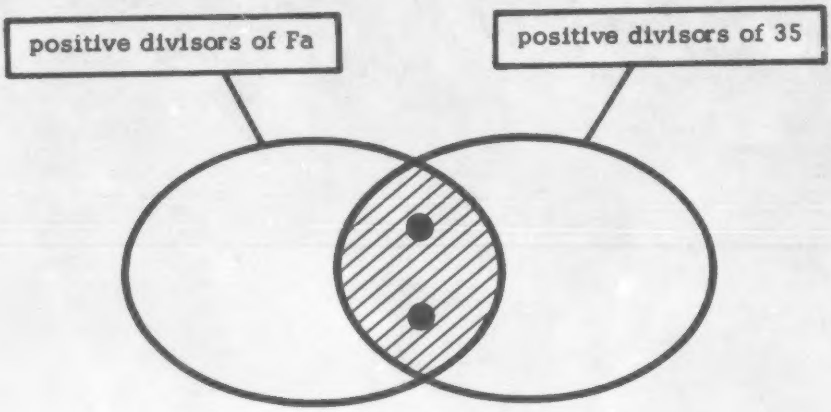
T: What information does this clue give us about Fa?

S: 3 is the greatest common divisor of 24 and Fa.

Let the students work independently testing the possibilities for Fa in this clue. The class should conclude that Fa is 3 or 15.

### Third clue

Draw this string picture on the board.



T: What new information does this string picture give us?

S: Fa and 35 have exactly two common divisors.

T: Who is Fa?

Allow a minute or two for the students independently to test the possibilities for Fa in this clue.

S: Fa is 15, because 15 and 35 have exactly two divisors in common — 1 and 5.

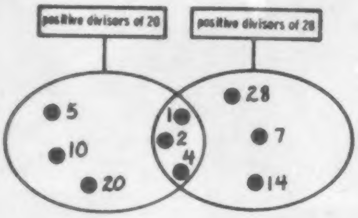
S: Fa cannot be 3, because 3 and 35 have only one common divisor — 1.

The students should conclude that Fa is 15.

Worksheets N44, N45, and N46 are available.

Name ______________________ N 44 *

Put all the positive divisors of 20 and of 28 in this string picture.


positive divisors of 20
positive divisors of 28
5
10
20
1
2
4
28
7
14

Complete each calculation.

20 ⊓ 28 = 4
10 ⊓ 28 = 2
35 ⊓ 28 = 7

---

Name ______________________ N 45 **

Zim is a secret whole number.

First clue

ZIM ⊔ 24 = 72

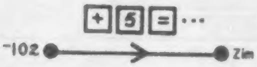
Zim could be 9, 18, 36, or 72.

Second clue

ZIM ⊓ 30 = 6

Zim could be 18, 36, or 72.

Third clue



Zim is 18.

---

Name ______________________ N 46 ***

Pom is a secret whole number.

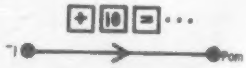
First clue

POM ⊓ 28 = 7

Find a pattern for the numbers that Pom could be.

Pom could be 7, 21, 35, 49, 63, 77, 91, 105, 119, 133, 147, and so on.

Second clue



Find a pattern for the numbers that Pom could be.

Pom could be 49, 119, 189, 259, 329, 399, and so on.

Third clue

In this list, Pom is the largest number smaller than 1000.

Pom is 959.

## 5.3 Divisibility by 9 #1

CAPSULE LESSON SUMMARY

Explore a new rule for a relation where the arrows mean that the ending number is the sum of the digits of the starting number. Use the arrow picture for this relation to begin looking at a test for divisibility by 9.

MATERIALS

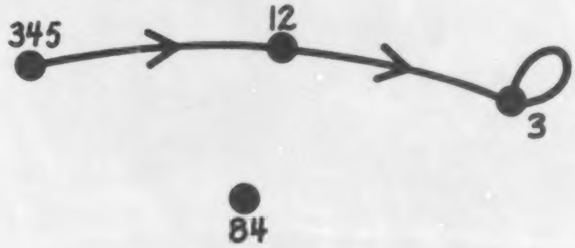
Teacher: Colored chalk

Student: Paper; colored pencils; Worksheets N47 and N48

### DESCRIPTION OF LESSON

#### Exercise 1

Draw this arrow picture on the board. Draw it to one side of the board so that it can be saved for later examination.



T: Do you remember THE VALENTINE MYSTERY?

Let the students tell as much of the story as they can. The class should recall that 0 invented a rule for sending valentines.

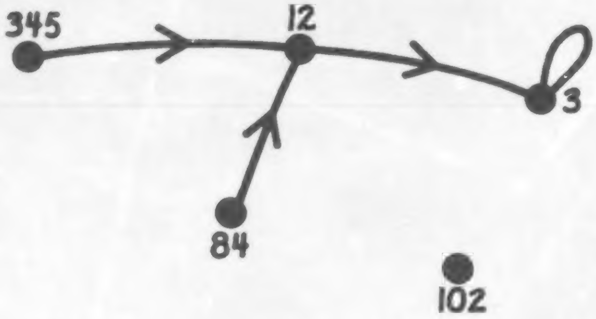
T: 0 has invented another rule for sending valentines. (Point to the arrow picture.) 345 sends a valentine to 12, who sends one to 3, who sends one to himself. To which number does 84 send a valentine?

NOTE: 0's new rule is: each whole number sends a valentine to the sum of its digits; for example, 3 + 4 + 5 = 12, so 345 sends a valentine to 12. Do not tell the class the rule at this time.

# N 13

Allow a short while for the students to study the picture before accepting guesses. Walk around the room and let students whisper their guesses in your ear. Acknowledge, aloud, any correct guess and deny any incorrect one. For example, "Yes, that's correct" or "No, 84 does not send a valentine to 3." After a while, ask a student who has the correct answer to announce that 84 sends a valentine to 12; or, if necessary, do so yourself.

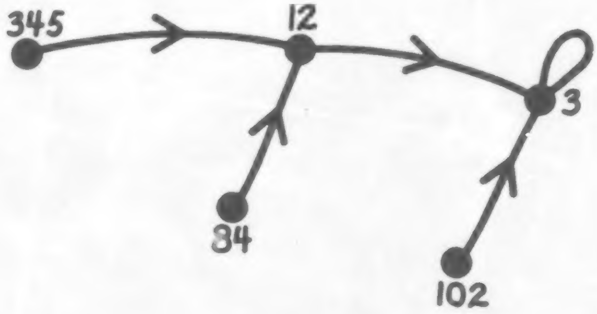
Draw a red arrow starting at 84 and ending at 12. Then add a dot for 102 to your picture.



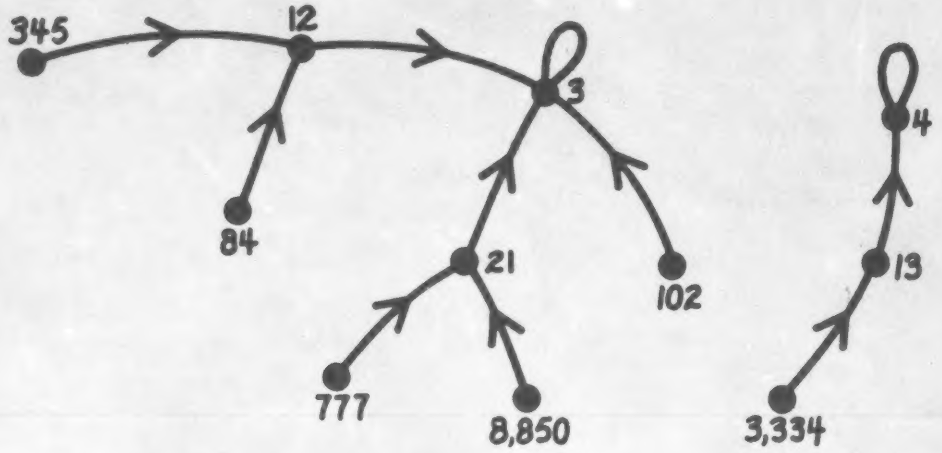
T: What about 102? Where would a red arrow starting at 102 end? To which number does 102 send a valentine?

S: 102 sends a valentine to 3.

Draw a red arrow starting at 102 and ending at 3.



Continue this "Guess My Rule" game a short while longer. Your picture might develop as in the illustration below.

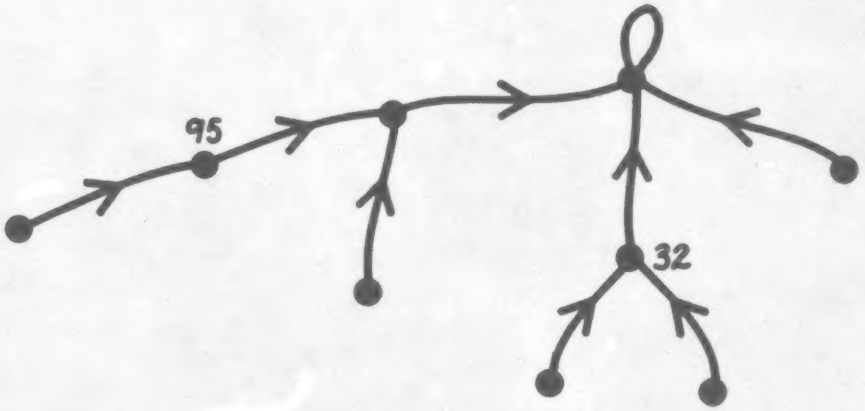


T: Can you guess 0's rule?

S: Each whole number sends a valentine to the sum of its digits; for example, 3,334 sends a valentine to 13 because 3 + 3 + 3 + 4 = 13.

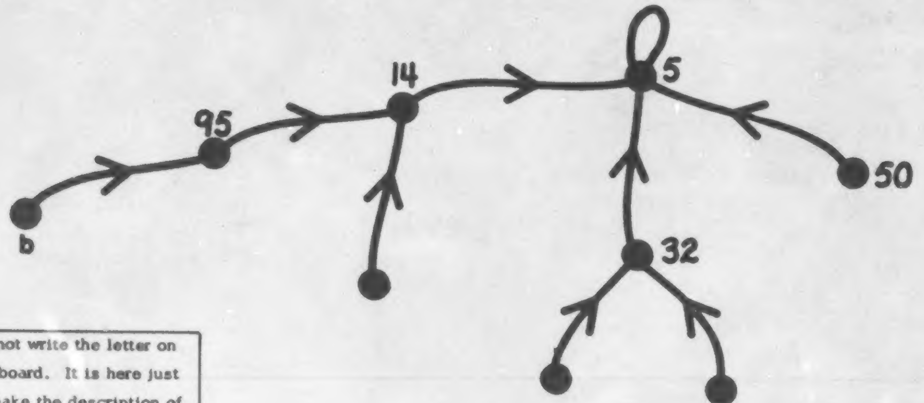
If the class does not guess the rule at this point, explain it yourself.

Draw the arrow picture below on the board. Instruct the students to copy it and to label the dots. Many solutions are possible, so encourage them to be creative.



# N 13

Suggest to faster workers that they add more arrows to their pictures and label the dots. After a while, invite students to label dots at the board. Suppose your picture looks like this:



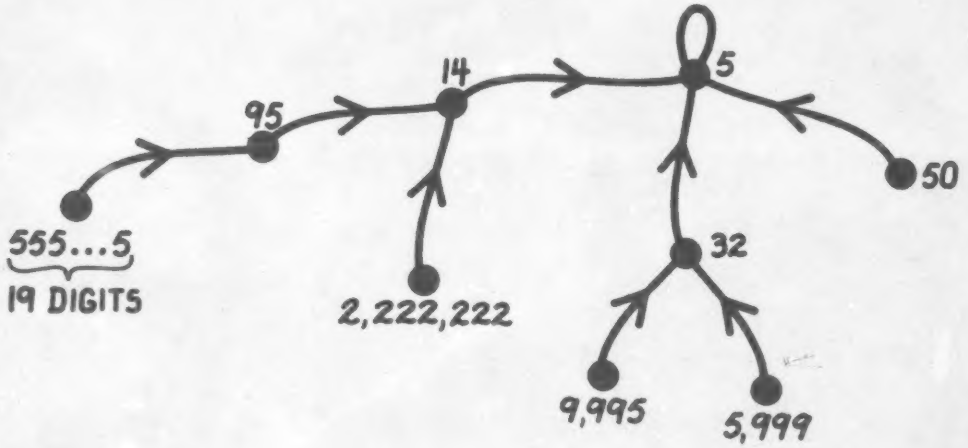
14
5
95
50
b
32

Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T (pointing to b): Which number could be here?

S: It must be a number whose digits add up to 95. It could be the number with nineteen 5's.

Label the dot and continue asking students to label dots.



14
5
95
50
555...5
19 DIGITS
2,222,222
32
9,995
5,999

T: How many valentines does each number receive?

S: Each number, except 0, will receive an endless number of valentines. For example, 5 will receive valentines from 5; 50; 500; 5,000; 50,000; 500,000; and so on.

S: 5 will also receive valentines from 14; 41; 104; 140; 410; 401; 1,004; 1,040; and so on.

T: Very good. What about 0? How many valentines does 0 receive?

S: 0 receives exactly one valentine — from 0.

T: How many valentines does each whole number send?

S: Exactly one valentine.

T: Good. So each whole number sends exactly one valentine and, with the exception of 0, each whole number receives an endless number of valentines.

Distribute Worksheets N47 and N48 for independent work. Allow enough time that most students are able to finish N47 and to begin on N48.

Exercise 2

T: Look at the arrow pictures on the worksheets and on the board. Are there any arrows which would connect these pieces? (No) How many pieces would there be in a full picture for 0's rule?

S: There would be ten pieces. A piece with just 0 in it; a piece with 1 in it, with 2, with 3, and so on to 9.

T: Very good. Let's look at the pieces and see if we notice anything interesting. Look at the piece that has 9 in it (on N47). What do you notice about the numbers in this piece?

If no suggestions are forthcoming, lead the discussion as follows.

T: Give me some other numbers that belong to this piece.

S: 63, because 6 + 3 = 9.

S: 405, because 4 + 0 + 5 = 9.

List numbers that the students suggest on the board. For example,

63, 405, 54, 18, 99, 81, 27, 999, 36, ...

S: Each number is a multiple of 9.

T: So the numbers in this piece of the arrow picture with 9 in it are multiples of 9. Look at the piece with 1 in it (on N48). What do you notice about the numbers in this piece?

If necessary, ask the class for other numbers that could be in the picture and list those numbers on the board. For example,

10; 64; 55; 100; 9,999,991; 1,000; 37; ...

S: Each number is 1 more than a multiple of 9. For example, 64 = 63 + 1; 9,999,991 = 9,999,990 + 1; and 37 = 36 + 1.

T: Good. Now, look at the piece with 3 in it (on the board). What do you notice about the numbers in this piece?

In a manner similar to the above, the students should conclude that the numbers in the piece of the arrow picture with 3 in it are 3 more than multiples of 9.

T: We have looked closely at three pieces of the arrow picture. What about the other pieces? What can we say about the numbers belonging to the other pieces?

Let the class consider this question for a few minutes.

S: Numbers which belong to the same piece are each a multiple of 9 plus the same number. For example, the numbers belonging to the piece with 7 in it are 7 more than multiples of 9.

If you feel it is appropriate, check a few additional numbers.

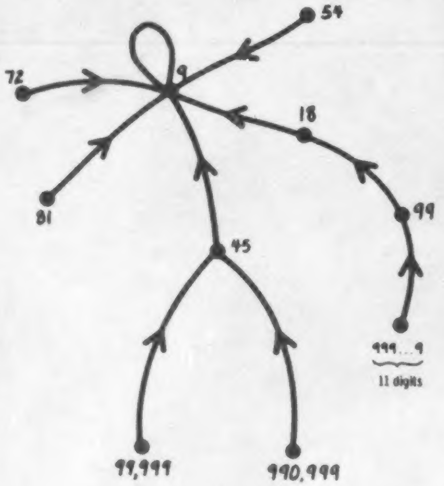
T: How can we explain this result?

Let students express their opinions, but do not worry if no explanations are forthcoming. This result will be examined again in Lesson N5.4.

Name ____________________

N 47 *

Label the dots. Many solutions are possible.


54
72
9
18
81
99
45
999...9
11 digits
99,999
990,999

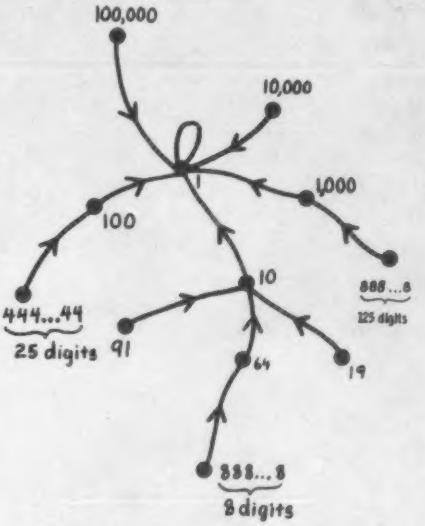
(Many solutions are possible.)

---

Name ____________________

N 48 **

Label the dots. Many solutions are possible.


100,000
10,000
1
1,000
100
10
888...8
125 digits
444...44
25 digits
91
64
19
888...8
8 digits

(Many solutions are possible.)

## 5.4 Divisibility by 9 #2

CAPSULE LESSON SUMMARY

Fill in the square with the smallest possible whole number for number stories such as "356 = △ + □ " where " △ " must be for some multiple of 9. Relate this activity to the arrow picture for 0's rule in Lesson N5.3, and find a test for deciding whether or not a number is a multiple of 9 or of 3. (A number is a multiple of 9 (3) if the sum of its digits is a multiple of 9 (3).) Provide missing digits so that numbers are multiple of 9.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

NOTE: This exercise will examine number stories such as:

$$10 = \triangle + \square$$

where some multiple of 9 goes in the triangle and some whole number goes in the square. It may be necessary for you frequently to remind the students that " △ " is always for some multiple of 9.

Write this number story on the board.

$$10 = \triangle + \square$$

T: Some multiple of 9 goes in " △ " and some whole number goes in " □ "; 10 is equal to some multiple of 9 plus some whole number. What is the smallest whole number that could be in " □ "?

# N 15

**S:** 1, because 10 = 9 + 1.

**T:** Yes, 10 is 1 more than a multiple of 9.

Write "1" in the square and consider this new story.

$$10 = \triangle + \boxed{1} \qquad \triangle\text{: multiple of 9}$$

$$40 = \triangle + \boxed{\phantom{0}}$$

**T:** 40 is equal to some multiple of 9 plus some whole number. What is the smallest whole number that could be in " $\boxed{\phantom{0}}$ "?

**S:** 4, because 40 = 36 + 4.

**T:** Yes. Do you see a way to solve this problem using the first number story?

**S:** Multiply all the numbers in the first story by 4. Four times a multiple of 9 plus 1 is some multiple of 9 plus 4.

Include this information on the board.

$$\begin{tikzcd}
10 \arrow[d, bend right, "4\times"'] & = \triangle + & \boxed{1} \arrow[d, bend left, "4\times"] \\
40 & = \triangle + & \boxed{4}
\end{tikzcd}$$

Continue this activity by asking the class to copy these number stories (see the next illustration) and to find for each one the smallest whole number that could be in " $\boxed{\phantom{0}}$ ".

$\triangle$: multiple of 9

$$100 = \triangle + \square$$
$$1{,}000 = \triangle + \square$$

$$70 = \triangle + \square$$
$$700 = \triangle + \square$$

$$300 = \triangle + \square$$
$$50 = \triangle + \square$$
$$6 = \triangle + \square$$
$$356 = \triangle + \square$$

Check the problems collectively by asking students to fill in the squares. Answers are given below.

$\triangle$: multiple of 9

$$100 = \triangle + \boxed{1}$$
$$1{,}000 = \triangle + \boxed{1}$$

$$70 = \triangle + \boxed{7}$$
$$700 = \triangle + \boxed{7}$$

$$300 = \triangle + \boxed{3}$$
$$50 = \triangle + \boxed{5}$$
$$6 = \triangle + \boxed{6}$$
$$356 = \triangle + \boxed{5}$$

Some students might answer that 356 is 14 more than a multiple of 9. For example:

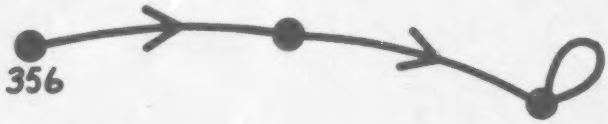
**S:** 356 = 300 + 50 + 6, so 356 is a multiple of 9 plus 3 + 5 + 6 or a multiple of 9 plus 14.

**T:** It is good that you used the previous work to help you, but are you sure that 14 is the smallest number that could be in " $\square$ "?

**S:** 14 = 9 + 5, so 356 is a multiple of 9 plus 5.

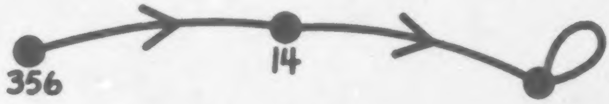
**T:** Yes, 5 is the smallest whole number that could be in " $\square$ ".

Draw this arrow picture on the board.



**T:** Do you remember 0's new rule[*] for sending valentines? To which number does 356 send a valentine?

**S:** 14, because 3 + 5 + 6 = 14.



**T:** And to which number does 14 send a valentine?

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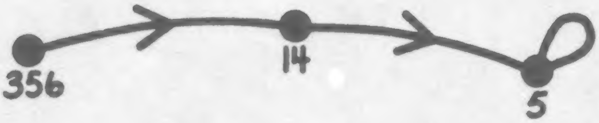
[*] See Lesson N5.3 for an explanation of the rule.

S: 5.



T: Do you see any connection between the arrow picture and the number story about 356?

S: 356 is 5 more than a multiple of 9 just like every number that would be in this part of the arrow picture for 0's rule.

Exercise 2

Write these numbers on the board.

5,784 6,786 40,781

257,796

T: Copy these numbers on your paper and indicate which are multiples of 9 and which are multiples of 3.

Allow a few minutes for individual work before discussing the problem collectively.

S: 5 + 7 + 8 + 4 = 24 so 5,784 is 24 more than a multiple of 9; 24 = 18 + 6 so 5,784 is 6 more than a multiple of 9.

Write this information on the board and remind the students that " △ " is for some multiple of 9.

△: multiple of 9

$$5{,}784 = \triangle + 6$$

T: So 5,784 is not a multiple of 9. Is it a multiple of 3?

S: Yes, each multiple of 9 is a multiple of 3 and 6 is a multiple of 3. A multiple of 3 plus a multiple of 3 is a multiple of 3.

T: Good. What about 6,786?

S: 6 + 7 + 8 + 6 = 27 and 27 is a multiple of 9, so 6,786 is a multiple of 9 and a multiple of 3.

Continue the discussion with 40,781 and 257,796. The students should conclude that 40,781 is 2 more than a multiple of 9 and not a multiple of 3, and that 257,796 is both a multiple of 9 and a multiple of 3.

T: Now we have an easy way to tell if a number is divisible by 9 or by 3. If the sum of the digits of a number is divisible by 9, then the number is divisible by 9. If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.

Exercise 3

Write this information on the board.

$$5\square 4{,}302 \qquad 2{,}37\square{,}428$$

$$87\square{,}769 \qquad 42{,}307{,}8\square 8$$

T: Fill in the missing digits of these numbers so that each one is a multiple of 9.

Provide several minutes for the students to work independently before discussing the problems collectively.

S: For the first problem, I added the digits; 5 + 4 + 3 + 0 + 2 = 14. Since 14 + 4 = 18 and 18 is a multiple of 9, the missing digit is 4.

Invite students to explain their solutions to the other problems. Answers are given below.

$5\boxed{4}4{,}302$ $2{,}37\boxed{1}{,}428$

$87\boxed{8}{,}769$ $42{,}307{,}8\boxed{4}8$

Write this information on the board.

Flip is 67□,34○

T: Flip is a multiple of 9. Find the numbers that Flip could be. Make a list on your paper.

After a short while for independent work, invite students to write the eleven numbers that Flip could be on the board. Flip could be:

$67\boxed{9}$,34⑦
$67\boxed{8}$,34⑧
$67\boxed{7}$,34⑨
$67\boxed{7}$,34⓪
$67\boxed{6}$,34①
$67\boxed{5}$,34②
$67\boxed{4}$,34③
$67\boxed{3}$,34④
$67\boxed{2}$,34⑤
$67\boxed{1}$,34⑥
$67\boxed{0}$,34⑦

# N 17

## 5.5 Exponents and Prime Factorization

CAPSULE LESSON SUMMARY

Introduce exponential notation using a Guess My Rule game for an operation $\star$ where $a \star b = a^b$. Write various numbers as products of prime numbers making use of the exponential notation. Solve a detective story where the clues involve prime divisors of the secret number, the relation $=_4$, and the hand-calculator relation $\boxed{-}\ \boxed{6}\ \boxed{=}\ldots$.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil

### DESCRIPTION OF LESSON

#### Exercise 1

Write this information on the board.

$$2 * 3 = 8$$
$$3 * 2 = 9$$
$$3 * 1 = 3$$
$$1 * 3 = 1$$
$$5 * 2 = \boxed{\phantom{0}}$$

T: I have a secret rule for $\star$ and have given you four clues about my rule. Can you guess my rule? Which number do you think goes in the box?

NOTE: The rule is: $a \star b = a^b$. The number in the box is 25 because $5 \star 2 = 5^2 = 5 \times 5 = 25$. Do not tell the class your rule at this time.

Suggest students whisper their guesses in your ear. Acknowledge, aloud, any correct guesses and deny any incorrect ones. For example, "Yes, that's correct" or "No, 10 is not the number". Let a student who guesses correctly tell the class that 25 is in the box, but do not give away the rule. If no one guesses correctly, put 25 in the box yourself.

Continue the activity with these problems (answers are in the boxes).

$$10 * 2 = \boxed{100}$$
$$4 * 2 = \boxed{16}$$
$$6 * 2 = \boxed{36}$$
$$8 * 2 = \boxed{64}$$
$$9 * 2 = \boxed{81}$$

T: Who can explain *2?

S: 10 * 2 = 100 because 10 x 10 = 100; 4 * 2 = 16 because 4 x 4 = 16; and so on.

S: For *2 you square the number.

T: Good. Let's look at *3.

Write these problems on the board and let the students whisper their guesses in your ear. Answers are in the boxes.

$$5 * 3 = \boxed{125}$$
$$3 * 3 = \boxed{27}$$
$$4 * 3 = \boxed{64}$$

T: Who can explain *3?

S: For *3 you multiply the number by itself three times; 5 * 3 = 5 x 5 x 5 = 125.

# N 17

Accept any reasonable explanation.

T: Very good. You know part of my rule. Can you guess my rule?

Write these problems on the board (answers are in boxes) and let the students guess.

$$2 * 4 = \boxed{16}$$
$$3 * 4 = \boxed{81}$$
$$10 * 3 = \boxed{1{,}000}$$
$$1 * 10 = \boxed{1}$$
$$10 * 1 = \boxed{10}$$
$$2 * 10 = \boxed{1{,}024}$$

T: Who can explain my rule?

S: 2 * 4 means multiply 2 by itself four times; $2 \times 2 \times 2 \times 2 = 16$.

Accept any reasonable explanations your students offer. As this is a rather difficult idea to verbalize, students might prefer to explain the rule at the board by writing the appropriate multiplication problem. For example,

$$2 * 4 = \boxed{16} = 2 \times 2 \times 2 \times 2$$
$$3 * 4 = \boxed{81} = 3 \times 3 \times 3 \times 3$$
$$10 * 3 = \boxed{1{,}000} = 10 \times 10 \times 10$$
$$1 * 10 = \boxed{1} = 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$$
$$10 * 1 = \boxed{10} = 10$$
$$2 * 10 = \boxed{1{,}024} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

T: As you can see it is sometimes quite cumbersome to write many multiplications, so there is a short-hand notation we will use.

Write this information on the board as you read it.

$$16 = 2 \times 2 \times 2 \times 2 = 2^4$$

T: For "2 x 2 x 2 x 2" write "$2^4$" (Read: two to the fourth power). Copy these problems and fill in the boxes.

Write these problems on the board (answers are in the boxes).

$$2^6 = \boxed{64}$$
$$3^5 = \boxed{243}$$
$$7^2 = \boxed{49}$$
$$4^3 = \boxed{64}$$

Check the problems collectively. Encourage students who have difficulty with this notation to first write the associated multiplication calculation.

Write this calculation on the board.

$$2^5 \times 2^3$$

T: What number is $2^5 \times 2^3$?

Allow a few minutes for students to consider the question.

S: 256.

## N 17

**T:** How did you get 256?

**S:** $2^5 = 32$ and $2^3 = 8$ and $32 \times 8 = 256$.

**S:** $2^5 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$ and $2^8 = 256$.

Write this information on the board.

$$
\begin{aligned}
&2^5 \times 2^3 = 2^8 = 256 \\
&3^2 \times 3^3
\end{aligned}
$$

**T:** What number is $3^2 \times 3^3$?

**S:** $3^2 \times 3^3 = 3^5$.

**S:** $3^2 \times 3^3 = 243$.

Summarize the students' comments by writing this information on the board.

$$
\begin{aligned}
&2^5 \times 2^3 = 2^8 = 256 \\
&3^2 \times 3^3 = 3^5 = 243
\end{aligned}
$$

**T:** How did you conclude that $3^2 \times 3^3 = 3^5$?

**S:** $3^2 = 3 \times 3$ and $3^3 = 3 \times 3 \times 3$, so $3^2 \times 3^3 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$.

**T:** Do you notice anything else interesting about these problems?

**S:** $2^5 \times 2^3 = 2^{5+3} = 2^8$.
$3^2 \times 3^3 = 3^{2+3} = 3^5$.

## Exercise 2

T: Write 36 as a product of prime numbers. You may use a prime number more than once. Try to use our short hand notation when a prime number occurs more than once in the product.

Give the students a minute to work independently and then call on a volunteer to write a product on the board.

$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$

Write these numbers on the board and instruct the students to write each number as a product of prime numbers. Feel free to adjust the numbers to the numerical level of your class.

42 24 70 100 124 84 200

Be sure to check the students work collectively. Solutions are given below.

$$42 = 2 \times 3 \times 7$$
$$24 = 2^3 \times 3$$
$$70 = 2 \times 5 \times 7$$
$$100 = 2^2 \times 5^2$$
$$120 = 2^3 \times 3 \times 5$$
$$84 = 2^2 \times 3 \times 7$$
$$200 = 2^3 \times 5^2$$

# N 17

**T:** Many of you have devised your own way to solve these problems. Let me show you one way to find the prime numbers.

Write "360" on the board.

**T:** Can you name a positive prime divisor of 360?

**S:** 2, 2 x 180 = 360 and 2 is prime.

Write this information on the board.

$$\begin{array}{r|l} 360 & 2 \\ 180 & \\ & \end{array}$$

**T:** Name a positive prime divisor of 180.

**S:** 2, 2 x 90 = 180 and 2 is prime.

$$\begin{array}{r|l} 360 & 2 \\ 180 & 2 \\ 90 & \\ & \end{array}$$

Continue this process until the last number in the left column is 1.

| | |
|---|---|
| 360 | 2 |
| 180 | 2 |
| 90 | 2 |
| 45 | 3 |
| 15 | 3 |
| 5 | 5 |
| 1 | |

**T:** Who can write 360 as a product of prime numbers?

| | |
|---|---|
| 360 | 2 |
| 180 | 2 |
| 90 | 2 |
| 45 | 3 |
| 15 | 3 |
| 5 | 5 |
| 1 | |

$$360 = 2^3 \times 3^2 \times 5$$

# N 17

## Exercise 3

Tell the students they will solve a detective story about Zog.

**First clue**

T: Zog is a secret whole number smaller than 100. The only positive prime divisors of Zog are 2 or 3.[#] Which numbers could Zog be?

S: Zog could be 4, because 2 x 2 = 4.

S: Zog could be 27, because 3 x 3 x 3 = 27.

S: Zog could be 6, because 2 x 3 = 6.

List the numbers that Zog could be on the board as the students suggest them. It is not necessary to get a complete list at this point. For example,

4, 27, 6, 12, 9, 3, 8, 2

Be sure to check any numbers that are disputed.

T: Most of you have found some of the numbers that Zog could be, but I'm not sure that we have found all the possibilities. Let's try to be more systematic in our search for these numbers.

---

[#] 2 may be a prime divisor of Zog or 3 may be a prime divisor of Zog or both 2 and 3 may be prime divisors of Zog. However, Zog may not have any other prime number as a divisor. You may need to expand your explanation of this condition on Zog with the class.

Draw this table on the board.

| $\times$ | $1$ | $3$ | $3^2$ | $3^3$ | $3^4$ |
|---|---|---|---|---|---|
| $1$ | | | | | |
| $2$ | | | | | |
| $2^2$ | | | | | |
| $2^3$ | | | | | |
| $2^4$ | | | | | |
| $2^5$ | | | | | |
| $2^6$ | | | | | |

T: Why do you think I stopped at $3^4$ and $2^6$?

S: $3^5$ and $2^7$ are both larger than 100.

T: Good. How can this table help us in our work?

Give the students time to consider the question.

S: In the first row we will have the numbers which have only 3 as a prime divisor. In the first column we will have the numbers which have only 2 as a prime divisor. In the rest of the table we will find numbers which have both 2 and 3 as prime divisors.

Invite students to complete the relevant portion of the table. Since Zog is smaller than 100 it is not necessary to fill in numbers larger than 100. (See the next illustration.)

| × | 1 | 3 | $3^2$ | $3^3$ | $3^4$ |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 9 | 27 | 81 |
| 2 | 2 | 6 | 18 | 54 | |
| $2^2$ | 4 | 12 | 36 | | |
| $2^3$ | 8 | 24 | 72 | | |
| $2^4$ | 16 | 48 | | | |
| $2^5$ | 32 | 96 | | | |
| $2^6$ | 64 | | | | |

The students should conclude that Zog could be any number (smaller than 100) that appears in the table.

Second clue

$$\text{Zog} =_4 38$$

T: What new information about Zog does this clue give us?

S: Zog is 2 more than a multiple of 4 just like 38.

S: The difference between Zog and 38 is a multiple of 4.

S: Zog is an even number but not a multiple of 4.

T: Who is Zog? Don't forget to use the table.

Let the students work independently for some time. Some might subtract and check to see if the difference is a multiple of 4. Others might draw a string picture like in Lesson N5.1. Still others might see some short-cuts.

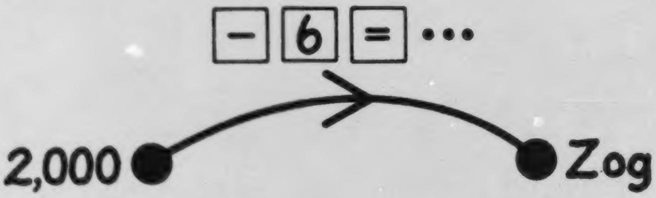
S: Since Zog is even, we can eliminate the numbers in the first row (3, 9, 27, 81).

S: 4, 12, 36 and all the numbers that appear below those numbers in the table can be eliminated because they are all multiples of 4, but Zog is 2 more than a multiple of 4. 38 is 2 more than a multiple of 4, so Zog must be also.

The students should conclude that Zog is 2, 6, 18, or 54.

Third clue

Draw this arrow picture on the board.



T: The third clue is given by this arrow picture. Who is Zog?

Let the students work independently before asking a volunteer to explain the solution.

S: If we press $\boxed{=}$ 300 times, we subtract 1,800; 2,000 - 1,800 = 200.
If we press $\boxed{=}$ 30 more times, we subtract 180; 200 - 180 = 20.
Then press $\boxed{=}$ six more times and subtract 18 to get 2. So Zog is 2.

The class should observe that this clue eliminates Zog being 6, 18, or 54 and hence conclude that Zog is 2.

# N 24

## 5.6 Prime Factor Distance

CAPSULE LESSON SUMMARY

Review the prime factor relation and find the prime factor distance between several pairs of numbers. Solve a detective story where the clues involve prime factor distance, the relation $=_3$ , and the hand-calculator relation $\boxed{+}\ \boxed{5}\ \boxed{=} \ldots$ .

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

### DESCRIPTION OF LESSON

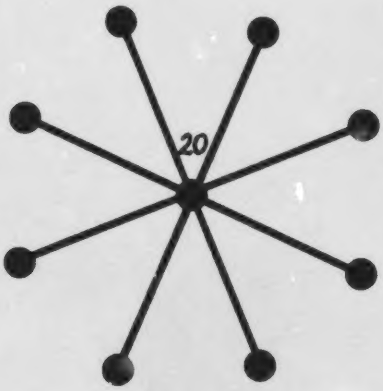
#### Exercise 1

Ask the class to name all the positive prime numbers smaller than 50. Write them on the board.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

Draw this cord picture on the board and indicate that red cords are for the prime factor relation.

prime factor relation



T: Do you remember the rule for the prime factor relation?

S: Two whole numbers are joined by a red cord if and only if one of the numbers equals a positive prime number times the other number.

T: Which numbers could be joined to 20 with a red cord?

S: 4, because 5 x 4 = 20 and 5 is a positive prime number.

S: 220, because 11 x 20 = 220 and 11 is a positive prime number.

S: 860, because 43 x 20 = 860 and 43 is a positive prime number.

Label the dots in your picture with the numbers suggested by the students. For three of the dots make more specific requests, but feel free to adjust the difficulty of your request to the numerical level of your class. For example,

T (pointing to a dot): Which is the largest number smaller than 500 that could be here?

S: 23 x 20 = 460 and 29 x 20 = 580. Since 29 is the next prime number after 23, 460 is the number.

T: Which is the largest number smaller than 1,000 that could be joined to 20 with a red cord?

S: 47 x 20 = 940 and 53 x 20 = 1,060. Since 53 is the next prime number after 47, 940 is the number.

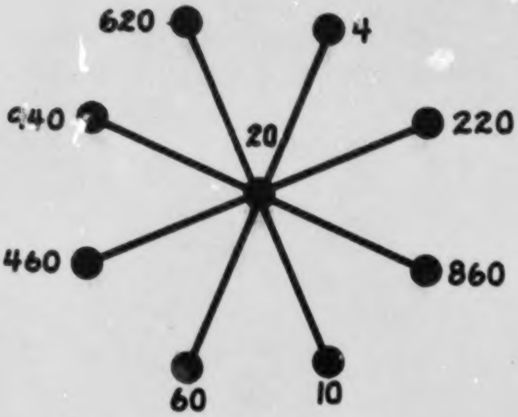
T: Which is the smallest number larger than 600 that could be in our picture?

# N 24

S: Since 31 is a prime number and 31 x 20 = 620 and 29 x 20 = 580, 620 is the number.

Your completely labeled picture might look like this.

prime factor relation



This activity could be repeated, if necessary, by replacing 20 with other numbers such as 12 or 15 or 6.

Exercise 2

Write this information on the board.

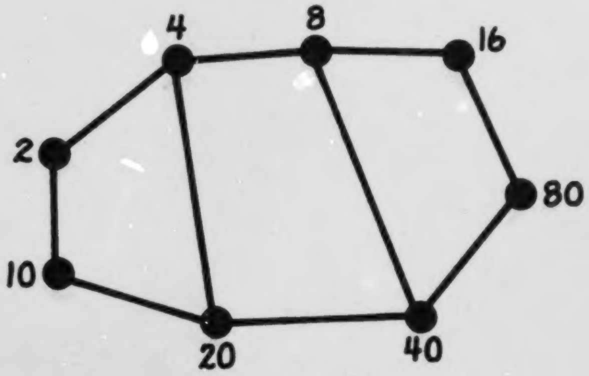
pfd (10, 16) =

prime factor relation

T: On your paper, use red cords to build a road between 10 and 16. Try to build a shortest road in order to find the prime factor distance.

Observe individual work for a few minutes. After many students have built a road between 10 and 16, invite several students to draw their solutions on the board. Do not insist that only shortest roads be drawn; many roads are possible. For example, your class might draw this picture.

prime factor relation



**T:** What is the length of the shortest road between 10 and 16?

**S:** 4.

**T:** Did anyone find a shorter road?

**S:** No.

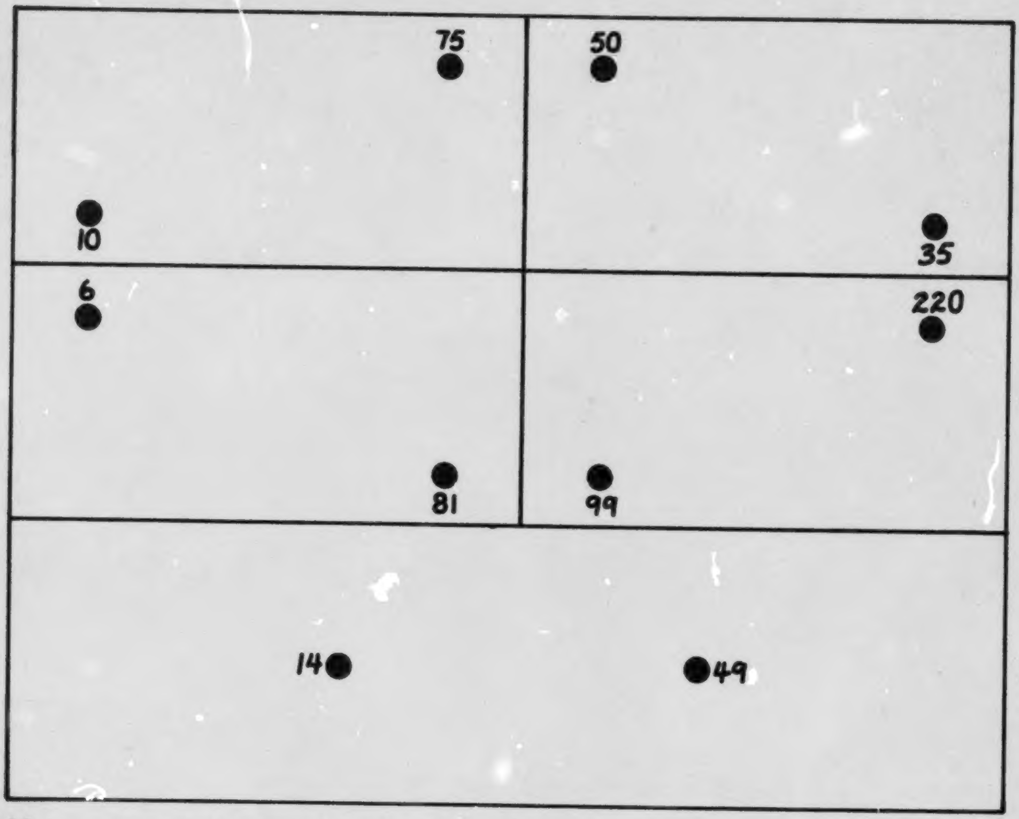
Complete "pfd (10, 16) = 4" and then write these problems on the board.

pfd (10, 75) =
pfd (6, 81) =
pfd (14, 49) =
pfd (50, 35) =
pfd (99, 220) =

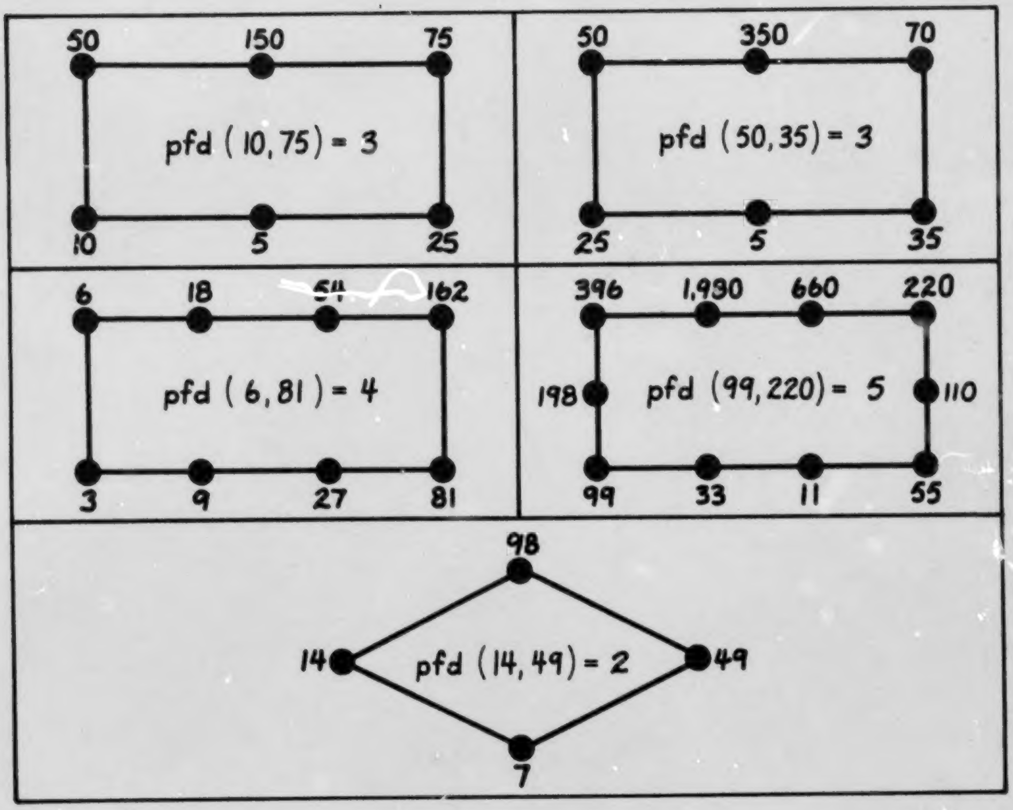
# N 24

T: For each of these problems, find the prime factor distance by drawing a shortest road between the two numbers.

As the students work independently, draw the appropriate pair of dots for each problem on the board.



After most students have build at least two roads, invite students to draw their roads on the board. Encourage students to draw more then one road for each problem. Do not insist that only shortest roads be drawn, but let students who find shorter roads add them to the picture. After a while, ask for the prime factor distance between each pair of numbers. Two possible shortest roads for each problem are presented in the next illustration. Many other roads are possible.


50
150
75
pfd (10, 75) = 3
10
5
25
50
350
70
pfd (50, 35) = 3
25
5
35
6
18
54
162
pfd (6, 81) = 4
3
9
27
81
396
1,980
660
220
198
pfd (99, 220) = 5
110
99
33
11
55
98
14
pfd (14, 49) = 2
49
7

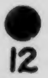
Exercise 3

Present this detective story about a secret whole number named "Blick".

First clue

Write this information on the board.

**Blick is a whole number less than 100**
**pfd (12, Blick) = 2**



## N 24

T: Blick is a secret whole number smaller than 100. The prime factor distance between Blick and 12 is 2. Which numbers could Blick be?

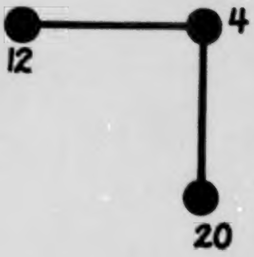
Adapt the following discussion to your students' suggestions.

S: Blick could be 20.

T: Why?

S: Draw a red cord from 12 to 4, because 3 x 4 = 12 and 3 is a positive prime number. Then draw a red cord from 4 to 20, because 5 x 4 = 20 and 5 is a positive prime number.

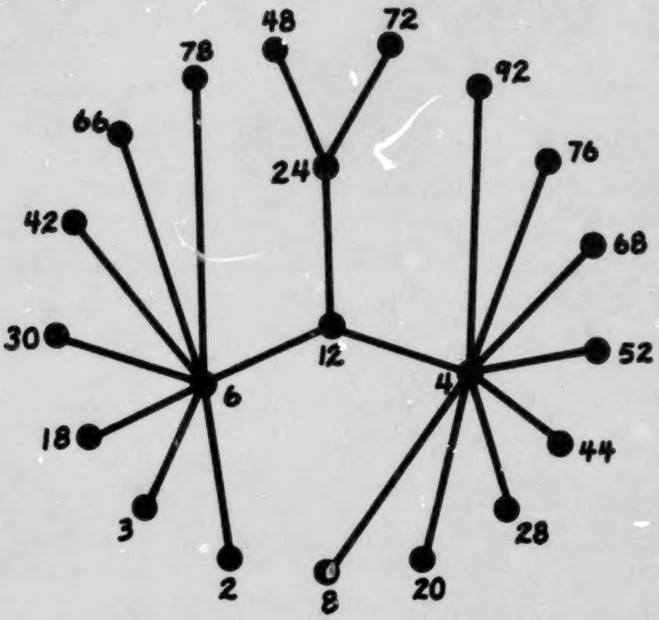
Draw the picture on the board.



NOTE: Blue dots are used to highlight the numbers that Blick could be.

After the class has found about four numbers that Blick could be, ask the students to copy the picture from the board and to find all the whole numbers less than 100 at a prime factor distance of 2 from 12. Let the students work independently for a while.

Invite students to complete the picture on the board. There are many ways to draw the paths to the seventeen numbers that Blick could be. A completed picture is shown in the next illustration.


48
72
78
92
66
24
76
42
68
30
12
52
6
4
18
44
3
28
2
8
20

T: Which numbers could Blick be?

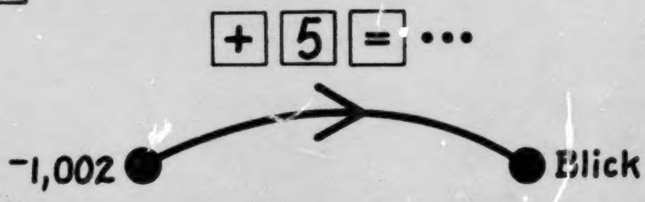
S: 2, 3, 18, 30, 42, 66, 78, 48, 72, 92, 76, 68, 52, 44, 28, 20, or 8.

Write the second and third clues on the board and let the students solve the detective story independently.

Second clue

$$\text{Blick} =_3 10$$

Third clue


+
5
=
...
⁻1,002
Blick

# N 24

When many students have solved the detective story, lead a brief discussion of the second and third clues.

T: What information does the second clue give us about Blick?

S: The difference between Blick and 10 is a multiple of 3.

S: 10 is 1 more than a multiple of 3 so Blick is also 1 more than a multiple of 3.

T: Yes, which numbers could Blick be?

S: Blick could be 28, because 28 - 10 = 18 and 18 is a multiple of 3.

S: 75 is a multiple of 3 and 76 is 1 more than 75, so Blick could be 76

S: 52 - 10 = 42 and 42 is a multiple of 3, so Blick could be 52.

The class should conclude from the second clue that Blick could be 28, 52, or 76.

T: What information do we get from the third clue?

S: If we press $\boxed{=}$ two hundred times, we add 1,000 ($^{-}1{,}002 + 1{,}000 = {}^{-}2$) so $^{-}2$ is on the display. If we press $\boxed{=}$ six more times, we add 30 ($^{-}2 + 30 = 28$) so 28 is on the display. If we continue pressing $\boxed{=}$, neither 52 nor 76 will appear. Blick is 28.

The class should conclude that Blick is 28.

## 5.7 Who is Gat?

**CAPSULE LESSON SUMMARY**

Examine characteristics of the sum of two multiples of 4, the sum of a multiple of 4 and a non-multiple of 4, and the sum of two non-multiples of 4. Solve a detective story in which the clues involve multiples of 4, patterns generated by moving one checker in a configuration on the Minicomputer, numbers which can be put on the Minicomputer with exactly one weighted checker, the relation $=_3$, and the hand-calculator relation $\boxed{-}\ \boxed{7}\ \boxed{=} \ldots$ .

**MATERIALS**

Teacher: Colored chalk
Student: Paper; pencil; Worksheets N49, N50, and N51

### DESCRIPTION OF LESSON

Exercise 1

Write this expression on the board.

$$\square + \triangle$$

T: If I put a multiple of 4 in the square and a multiple of 4 in the triangle, what can you tell me about the sum?

Let the class consider the question for a few minutes.

S: The sum is a multiple of 4. A multiple of 4 plus a multiple of 4 is a multiple of 4.

T: Let's try some examples. Give us a multiple of 4 to put in the square.

S: 8.

# N 28

**T:** Give us a multiple of 4 to put in the triangle.

**S:** 20.

**T:** Which number is 8 + 20?

**S:** 28.

**T:** Is 28 a multiple of 4?

**S:** Yes, because 7 x 4 = 28.

Illustrate this fact with several more examples.

Repeat this activity with other conditions on the numbers that go in the square and triangle. Two situations are outlined below. Be sure to illustrate these facts with several examples.

□ : Multiple of 4

△ : Non-multiple of 4

Sum : Non-multiple of 4

__Examples__

12 + 7 = 19

16 + 6 = 22

36 + 21 = 57

40 + 102 = 142

24 + 37 = 61

□ : Non-multiple of 4

△ : Non-multiple of 4

Sum : The sum can be either a multiple of 4 or a non-multiple of 4 depending on which non-multiples of 4 are chosen.

__Examples__

9 + 21 = 30 (Non-multiple of 4)

6 + 37 = 43 (Non-multiple of 4)

18 + 18 = 36 (Multiple of 4)

31 + 13 = 44 (Multiple of 4)

51 + 50 = 101 (Non-multiple of 4)

35 + 59 = 94 (Non-multiple of 4)

## Exercise 2

Present this detective story about a secret number named "Gat".

### First clue

T: Gat is a multiple of 4. Name some multiples of 4.

Call on students to name several multiples of 4.

T: On your paper, write some numbers between 635 and 655 that Gat could be.

S: Gat could be 636, 640, 644, 648, or 652.

T: Yes. Write the largest number smaller than 1,000 that Gat could be. Which number is it?

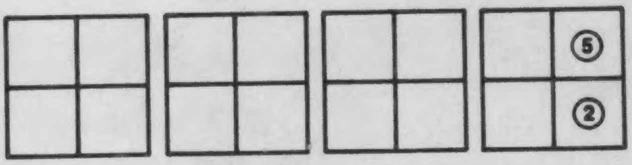
S: 996; because 1,000 is a multiple of 4 so 1,000 - 4 = 996 is the next smaller multiple of 4.

T: Which is the smallest number larger than 550 that Gat could be?

S: 552; because 548 is a multiple of 4 (500 and 48 are both multiples of 4) so 548 + 4 = 552 is the next larger multiple of 4.

### Second clue

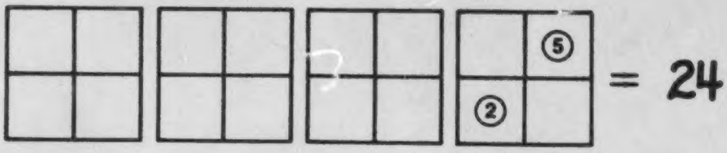
Display four Minicomputer boards with this configuration of checkers.

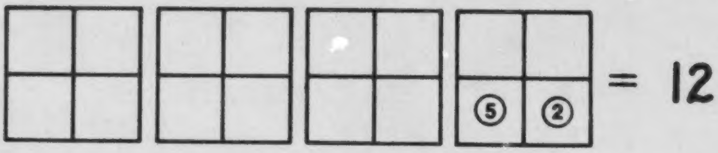


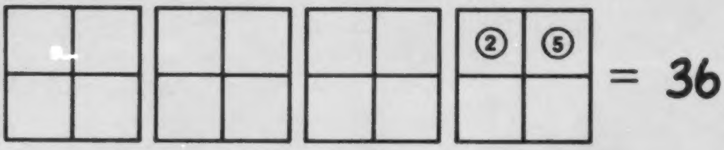
# N 28

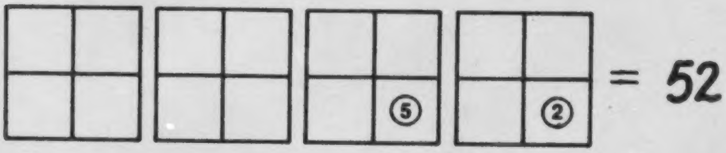
**T:** If you move one of these checkers to another square, you will get Gat. You may use more than just these four Minicomputer boards. Which numbers could Gat be?

Invite students to show possibilities for Gat by moving one checker on the Minicomputer. When necessary, remind a student that Gat is a multiple of 4. Some numbers that Gat could be are shown below.







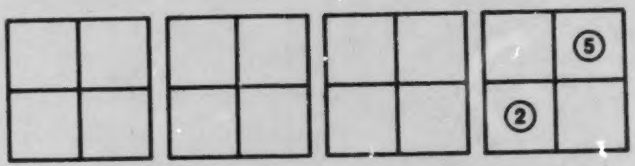


When it is obvious that most students understand the clue, continue as follows.

**T:** Let's try to be more systematic about finding numbers that Gat could be. Perhaps we will observe a pattern.

I will move the ② -checker to other squares. Each time the number is a multiple of 4, raise your hand.

Start by moving the ②-checker to the 2-square.



Since 24 is a multiple of 4, the students should raise their hands. Write "24" on the board to begin a list of possibilities for Gat. Continue in this manner moving the ②-checker from square to square. After moving the ②-checker to all the other squares (on the four displayed boards), your list of numbers for Gat should have at least eleven entries.

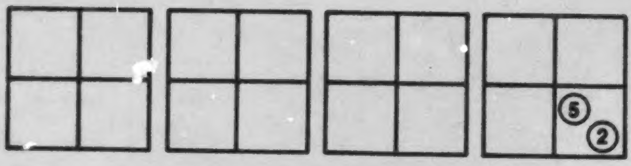
24
28
36
40
60
100
180
220
420
820
1,620
⋮

S: We could go on to other boards, but then no matter where we move the ②-checker we get a multiple of 4, because any multiple of 100 plus 20 is a multiple of 4.

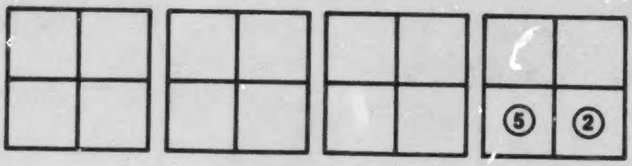
T: Now, I will move the ⑤-checker. Again raise your hand when we have a multiple of 4.

# N 28

Start by moving the ⑤ -checker to the 1-square...



...then to the 2-square.



Since 12 is a multiple of 4, the students should raise their hands. Write "12" on the board to begin a second list of possibilities for Gat. Continue moving the ⑤ -checker from square to square. After moving this checker to all the other squares (on the four displayed boards), you should have a second list with just two numbers that Gat could be.

12
52

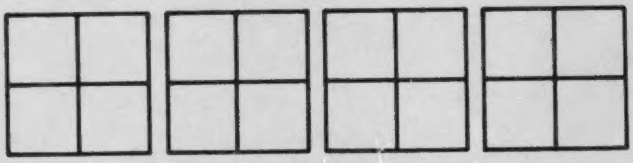
S: From now on no matter where we move the ⑤ -checker we will not get a multiple of 4 because we always add 2 to a multiple of 4.

The class should conclude that Gat could be any of the numbers in the two lists.

**Third clue**

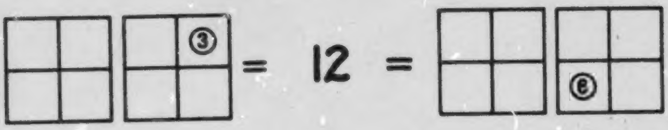
Display four Minicomputer boards and this set of weighted checkers.

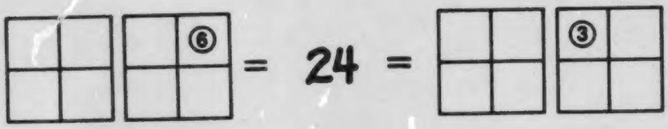
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

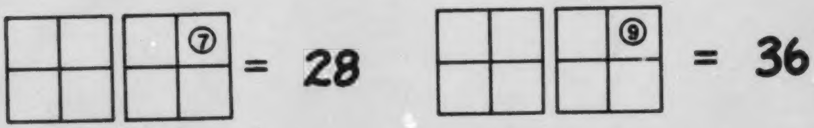


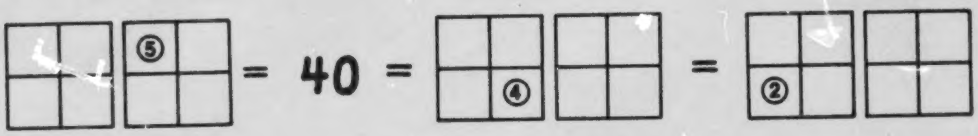
T: Gat can be put on this Minicomputer using exactly one of these checkers.

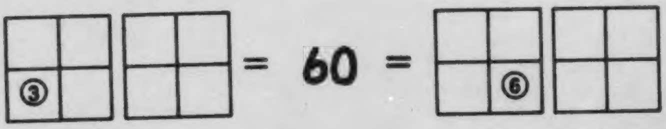
Ask students to place one checker on the Minicomputer to show possibilities for Gat. The class should conclude that Gat is one of the following numbers:

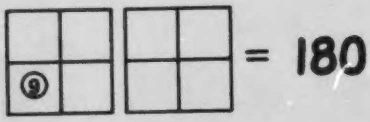












## Fourth clue

Write this information on the board.

$$\text{Gat} =_3 100$$

**T:** What new information does this clue give us about Gat?

**S:** The difference between Gat and 100 is a multiple of 3.

**S:** Gat is 1 more than a multiple of 3.

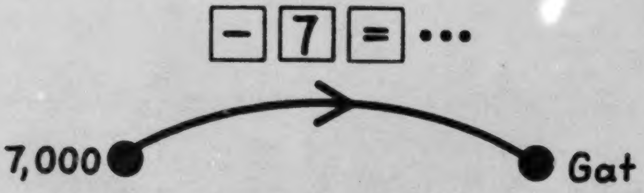
**T:** Which numbers could Gat be?

**S:** Gat could be 40, because 100 - 40 = 60 and 60 is a multiple of 3.

**S:** Gat could be 28, because 27 is a multiple of 3 and 28 is 1 more than a multiple of 3.

## Fifth clue

Draw this arrow picture on the board.



**T:** The fifth clue is given by this arrow picture. Who is Gat?

Let the students work independently with this clue for a few minutes.

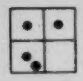
S: If we press $\boxed{=}$ one thousand times, we subtract 7,000 and 0 is the result. By adding sevens we can display 28 but not 40. Gat is 28.

S: The clue tells us that Gat is a multiple of 7. 28 is a multiple of 7 and 40 is not, so Gat is 28.

Worksheets N49, N50, and N51 are available.

Name ____________________ N 49 *

On each Minicomputer, add one regular checker to get a multiple of 4.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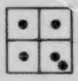
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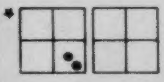
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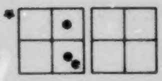
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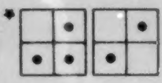
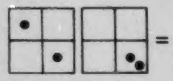

= 76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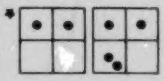
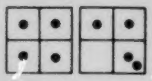

= 136


= 164

(For the starred problems, more than one solution is possible.)

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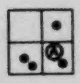
Name ____________________ N 50 **

On each Minicomputer, add one regular checker and one negative checker to get a multiple of 4. Many solutions are possible.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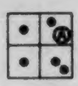
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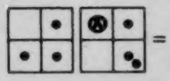
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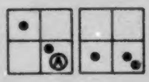
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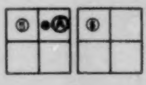
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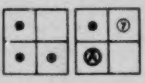
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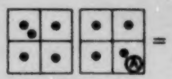
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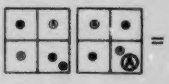
= 36

(Many solutions are possible.)

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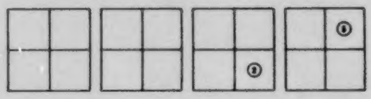
Name ____________________ N 51 ***



Glip is a multiple of 3 and can be shown on the Minicomputer by moving exactly one of these checkers to another square.

List all the numbers that Glip could be.

30; 60; 120; 420; 1,020; 4,020;

10,020; 40,020; 24; 36; 180; 1,620;

16,020.

## 5.8 Divisibility

CAPSULE LESSON SUMMARY

Find all the positive divisors from 2 to 10 of some selected numbers. Review tests for deciding whether or not a given number is divisible by 2, by 3, by 4, by 5, by 6, by 8, by 9, or by 10. Consider the numbers which can be formed by randomly arranging four digits and find the probabilities of getting numbers divisible by 2, by 3, by 4, by 8, by 9, by 5, and by 6.

MATERIALS

Teacher: None
Student: Paper; pencil; Worksheets N52, N53, and N54

### DESCRIPTION OF LESSON

Exercise 1

Write this number on the board.

24,630

T: Name a positive divisor of 24,630.

S: 10 is a positive divisor of 24,630 because 10 x 2,463 = 24,630.

T: Yes, 24,630 is divisible by 10. Name another positive divisor of 24,630.

Mix the use of the phrases "is a positive divisor of" and "is divisible by" throughout this lesson so that your students will become accustomed to using both expressions.

S: 24,630 is divisible by 2 because 24,630 is an even number.

T: Good. ... another positive divisor of 24,630?

**S:** 3. If you add the digits in 24,630 (2 + 4 + 6 + 3 + 0 = 15), you get 15. Since 15 is a multiple of 3, 24,630 is a multiple of 3.

If your students do not remember this test from a previous lesson, remind them of it. Be sure they understand it is a test for divisibility by 3 or by 9 and not a general divisibility test.

Summarize the information you have about 24,630 on the board.

**24,630** is divisible by

**10   2   3**

**T:** We know that 24,630 is divisible by 10, 2, and 3. There are other numbers between 1 and 10 that are divisors of 24,630. Can you find them?

**S:** Since 10 is a divisor of 24,630, 5 also is a divisor of 24,630.

**S:** Since 2 and 3 are divisors of 24,630, 6 must be a divisor of 24,630.

**T:** Yes, if a number is divisible by 2 and by 3, then it is divisible by 6.

Include 5 and 6 in your list of divisors of 24,630.

Ask the students to name some divisors of 25,371.

**S:** 2 + 5 + 3 + 7 + 1 = 18 and 18 is a multiple of 3, so 3 is a divisor of 25,371.

**S:** 18 is also a multiple of 9, so 25,371 is divisible by 9.

Record this information on the board.

25,371 is divisible by

3 9

T: What about 33,376? Which numbers divide 33,376?

S: 33,376 is an even number, so 33,376 is divisible by 2.

Write this information on the board.

33,376 is divisible by

2

S: 33,376 = 33,300 + 76. Since 33,300 and 76 are both multiples of 4, 4 is a divisor of 33,376.

S: 33,376 is also divisible by 8, because 376 ÷ 8 = 47 and 33,000 is divisible by 8.

NOTE: 1,000 is divisible by 8 and any multiple of 1,000 is divisible by 8, so a multiple of 1,000 plus a multiple of 8 is also a multiple of 8.

Record this information on the board.

33,376 is divisible by

2 4 8

Be prepared during the discussion to review any particular test for divisibility that the students do not recall.

Exercise 2

Write this information on the board.

4, 3, 0, 5

T: On your paper, write any number you wish using each of these digits exactly once. Decide if your number is divisible by 2, 3, 4, 5, 6, 8, 9, and/or 10.

Give the students several minutes to work independently and help those who need some additional attention.

Ask for a number and let the class find which numbers from 2 to 10 are positive divisors of the suggested number. For example.

5,304 is divisible by 2,3,4, and 6.

3,045 is divisible by 3 and 5.

Repeat this activity with one or two more suggested numbers.

T: Let's pretend that each of these digits is written on a slip of paper and that the four slips of paper are randomly arranged to get a four-digit number. What is the probability that we get a number which is divisible by 2?

Allow some time for the class to consider your question.

S: Divisibility by 2 is determined by a number's last digit. In this case, the last digit may be 0 or 3 or 4 or 5. If the last digit is 0 or 4, the number is even and divisible by 2. If the last digit is 3 or 5, the number is odd and not divisible by 2. There are two chances out of four that the number is divisible by 2.

Do not expect a student to give as detailed an explanation as the above. Instead direct the discussion along the lines outlined in the above response. Try to involve as many students as possible in such a discussion.

The class should conclude that the probability of the number being divisible by 2 is $\frac{1}{2}$.

T: What is the probability that we get a number which is divisible by 3?

S: 4 + 3 + 0 + 5 = 12 and 12 is a multiple of 3, so no matter how the digits are arranged the sum of digits will always be 12. The number will always be divisible by 3. The probability is 1.

T: Good. What is the probability that we get a number which is divisible by 4?

A method of solution to this problem might not be forthcoming. Help the class to focus on the last two digits of the number.

T: How can we tell if a number is divisible by 4? Remember, multiples of 100 are divisible by 4.

S: Look only at the last two digits. If that number is divisible by 4, then the number itself is divisible by 4.

T: How does this help us to find the probability that we get a number which is divisible by 4?

S: We can look at all possibilities for the last two digits.

The class should find twelve possible two-digit ending numbers. List these numbers on the board as they are suggested.

50 53 54 45

40 43 34 35

30 03 04 05

S: If the number ends in "40" or "04", then it will be divisible by 4. The probability is $\frac{2}{12}$ or $\frac{1}{6}$.

T: What is the probability that the number is divisible by 8?

After the discussion for divisibility by 4, some students may suggest that since 1,000 is a multiple of 8, only the possibilities for the last three digits need be examined. There are twenty-four possibilities; however, the work can be shortened if you recall that a number divisible by 8 must also be divisible by 4. Keeping this in mind, it is only necessary to look at these four of the twenty-four possibilities for the last three digits.

540 504

340 304

Two of these numbers, 504 and 304, are divisible by 8, so the class should conclude that the probability of getting a number which is divisible by 8 is $\frac{2}{24}$ or $\frac{1}{12}$.

Continue this exercise by asking the students to calculate the probability that the number is divisible by 9, by 5, and by 6. A brief explanation of the analysis of these situations is provided below.

Divisibility by 9: Since 4 + 3 + 0 + 5 = 12 and 12 is not a multiple of 9, any number whose digits are 4, 3, 0, and 5 will not be divisible by 9. The probability is 0.

Divisibility by 5: A number is divisible by 5 if and only if its last digit is 0 or 5. There are two chances out of four that the last digit of the randomly arranged number will be 0 or 5. The probability is $\frac{2}{4}$ or $\frac{1}{2}$.

Divisibility by 6: A number is divisible by 6 if and only if it is divisible by both 2 and 3. Previously, we found that the randomly arranged number would always be divisible by 3. There are two chances out of four of getting a number which is divisible by 2. So, the probability of getting a number which divisible by 6 is $\frac{2}{4}$ or $\frac{1}{2}$.

Worksheets N52, N53, and N54 are available.

Name ____________________ N 52 *

Zot is a secret number.

**First clue**

Zot is an even number and Zot's name can be completed by putting exactly one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in the box.

ZOT = 96,73☐

Zot could be 96,730, 96,732, 96,734, 96,736, or 96,738.

**Second clue**

Zot is divisible by 3.

Zot could be 96,732 or 96,738.

**Third clue**

Zot is not divisible by 4.

Zot is 96,738.

---

Name ____________________ N 53 **

Write a "T" for each true number story. Four true number stories are located for you.

| Is Divisible By | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 5,560 | T |  | T | T |  | T |  | T |
| 320,787 |  | T |  |  |  |  | T |  |
| 82,386 | T | T |  |  | T |  | T |  |
| 11,881 |  |  |  |  |  |  |  |  |
| 1,080 | T | T | T | T | T | T | T | T |
| 118,852 | T |  | T |  |  |  |  |  |
| 12,115 |  |  |  | T |  |  |  |  |
| 21,060 | T | T | T | T | T |  | T | T |
| 102,246 | T | T |  |  | T |  |  |  |
| 10,072 | T |  | T |  |  | T |  |  |
| 394,581 |  | T |  |  |  |  |  |  |

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Name ____________________ N 54 ***

FLOP = 56,08☐

Flop's name can be completed by choosing at random exactly one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 to put in the box.

What is the probability that:

Flop is divisible by 2? $\frac{1}{2}$

Flop is divisible by 3? $\frac{3}{10}$

Flop is divisible by 4? $\frac{3}{10}$

Flop is divisible by 5? $\frac{1}{5}$

Flop is divisible by 6? $\frac{1}{5}$

Flop is divisible by 8? $\frac{1}{5}$

Flop is divisible by 9? $\frac{1}{10}$

# 6

# DIVISION

The lessons in this chapter reinforce and extend the experiences of Part IV of the CSMP Intermediate Grades curriculum relative to situations involving the standard division algorithm and specialized division algorithms. A new abacus (with usual decimal place value in the bottom row of squares and a seven times rule going from a bottom square to a top square) introduces a special algorithm for division by 7. Problems related to sharing an amount of money equally among some specified number of people present the standard division algorithm with decimals.

## CAPSULE LESSON SUMMARIES

| | |
|---|---|
| Section 6.1: | Introduce a new abacus with usual decimal place value in the bottom row of squares and a seven times rule going from a bottom square to a top square. Explore the abacus by decoding numbers on it and by making plays (according to its rules) with both regular and weighted checkers. Observe the ease of putting seven times a given number on the abacus.<br>(Worksheets N55, N56) |
| Section 6.2: | Review the rules of the new 7-abacus introduced in Section 6.1 and practice decoding numbers and making plays on it. Investigate the effect of moving or trading certain checkers in a configuration on the abacus. Change a number on the abacus by a specified amount moving exactly one checker. Examine special configurations for numbers which allow the abacus to provide an algorithm for dividing by 7.<br>(Worksheets N57, N58) |

Section 6.3: Solve division problems suggested by situations in which a given amount of money is shared equally among some specified number of people. Present a form of the standard division algorithm with decimals. (Worksheets N59, N60)

Section 6.4: Review the rules and plays of the 7-abacus introduced in Section 6.1. Practice decoding numbers on the abacus and using plays to make the numbers easier to read. Investigate the effect of moving or trading certain checkers in a configuration on the abacus. Change a number on the abacus by a specified amount moving exactly one checker. Examine special configurations for numbers which allow the abacus to provide an algorithm for dividing by 7. (Worksheets N61, N62)

Section 6.5: Solve division problems suggested by situations in which a given amount of money is shared equally among some specified number of people. Present a form of the standard division algorithm with decimals.

Section 6.6: Review the rules and plays of the 7-abacus introduced in Section 6.1. Practice decoding numbers on the abacus and using plays to make numbers easier to read. Use the abacus to solve some division problems which involve sharing amounts of money equally among seven people. Do the calculation "1 ÷ 7" on the abacus to find that $\frac{1}{7} = 0.\overset{*}{\overline{142857}}$.

## 6.1 A 7-Abacus #1

CAPSULE LESSON SUMMARY

Introduce a new abacus with usual decimal place value in the bottom row of squares and a seven times rule going from a bottom square to a top square. Explore the abacus by decoding numbers on it and making plays (according to its rules) with both regular and weighted checkers. Observe the ease of putting seven times a given number on the abacus.

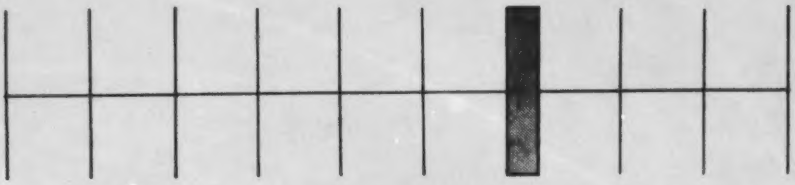
MATERIALS

Teacher: Minicomputer checkers (optional)
Student: Paper; pencil; Worksheets N55 and N56

DESCRIPTION OF LESSON

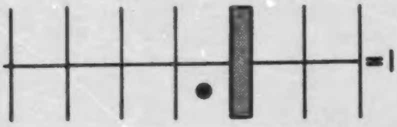
Exercise 1

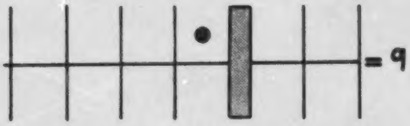
Draw this abacus on the board.

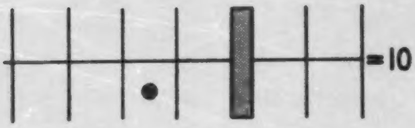


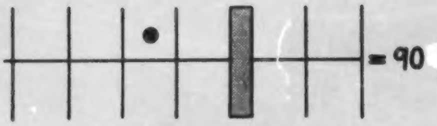
T: Do you remember the abacus that Nabu invented to solve "division by 9" problems?

Let the students recall as much about Nabu's 9-abacus as they can and then briefly review the values of the squares on this abacus.

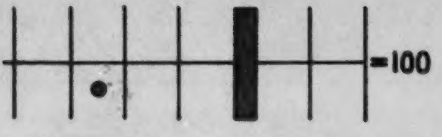


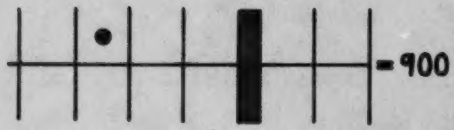


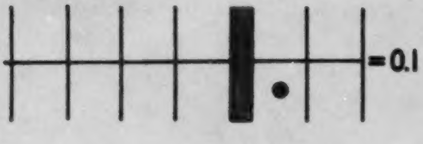


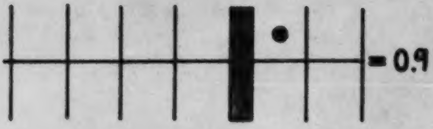


# N 10



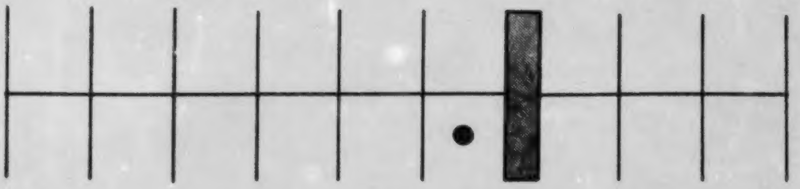






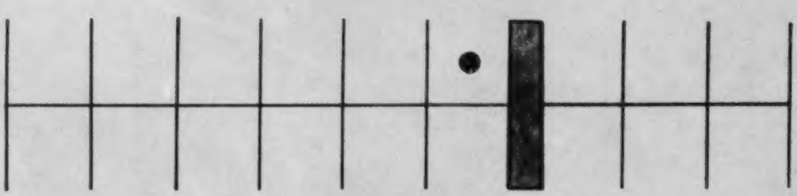
T: Nabu was so excited by how this abacus helped him solve "division by 9" problems that he invented other abaci to solve other division problems. Today we will study one of them, a 7-abacus.

Place a checker (use a regular Minicomputer checker if your board is magnetic, otherwise draw the checker with chalk) on the 1-square.



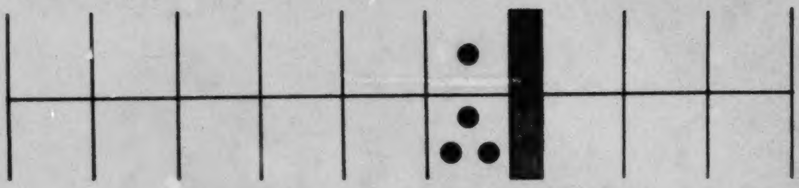
T: This is the number 1.

Move the checker to the square directly above the 1-square.



T: This is 7.

Add three checkers to the 1-square.

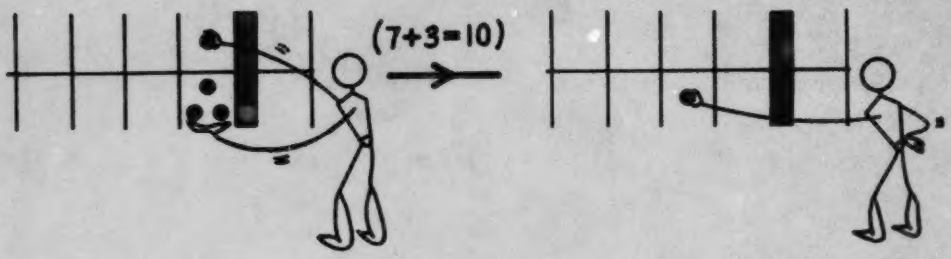


**T:** What number is this?

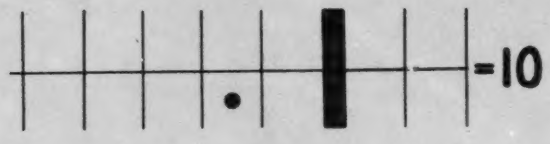
**S:** 10, 7 + 3 = 10.

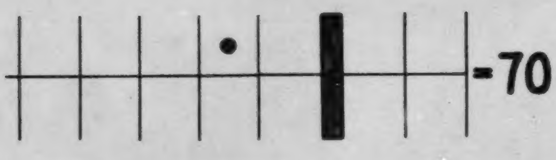
**T:** There is another way to put 10 on the abacus with just one checker. Do you know how to do it?

Invite a student to make the play illustrated below, or demonstrate it yourself.

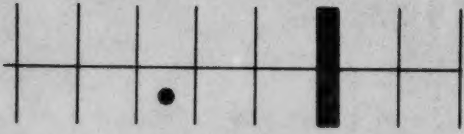


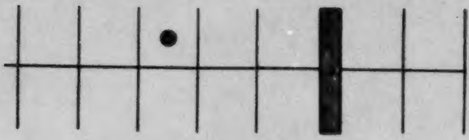
Continue the activity with these configurations.

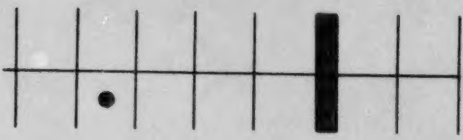


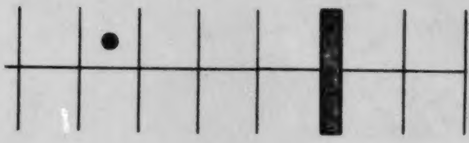


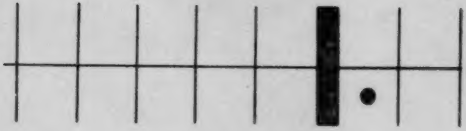
**N 10**

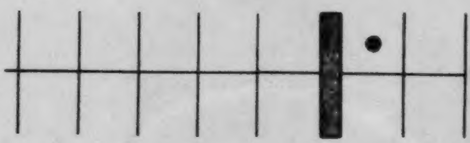

= 100

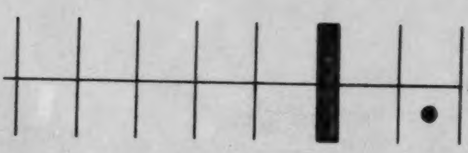

= 700

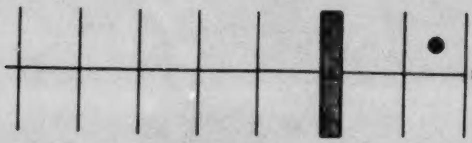

= 1,000


= 7,000

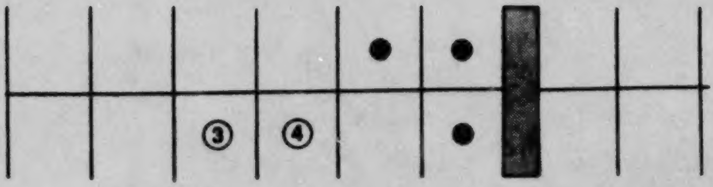

= 0.1


= 0.7


= 0.01

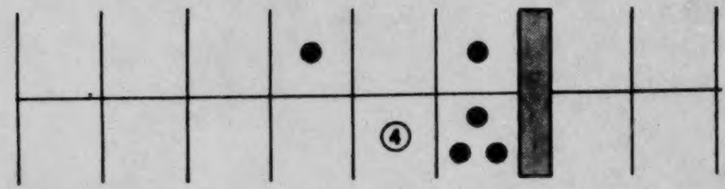

= 0.07

Display this configuration of checkers.



T: Which number is this?

S: 3,478; 3,000 + 400 + 70 + 8 = 3,748.



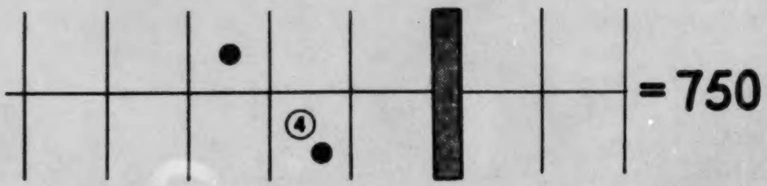
T: Which number is this?

S: 750; 700 + 40 + 10 = 750.

T: Good. Who can show the same number with fewer checkers?

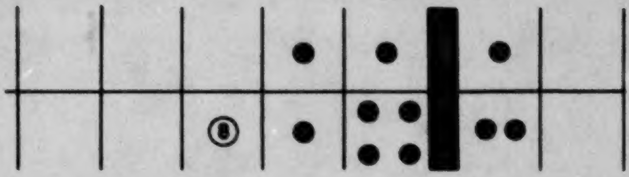
Invite a student to make a play.

S: 7 + 3 = 10.



# N 10

Display this configuration of checkers.



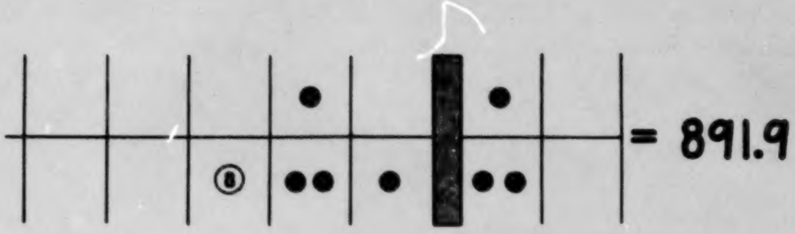
**T:** Write this number on your paper.

Allow a minute or two for students to decode the number.

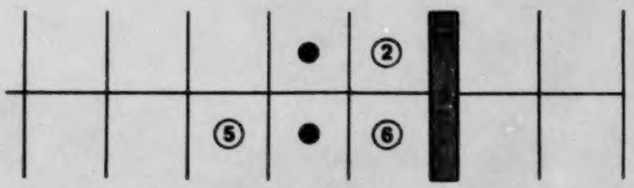
**S:** 891.9.

**T:** Yes. Who can show the same number with fewer checkers?

**S:** 7 + 3 = 10.



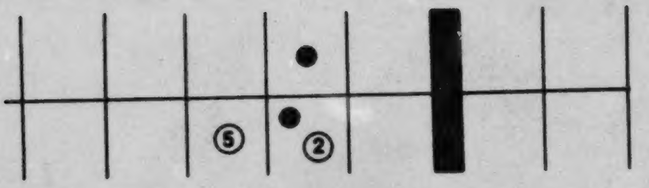
Display this configuration of checkers and ask the students to decode it.



**S:** 600.

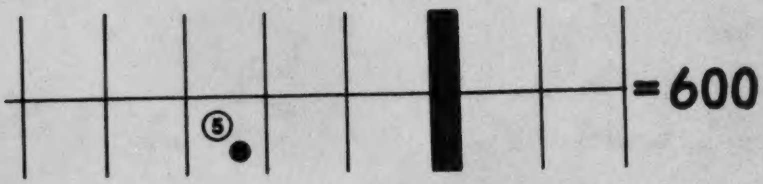
**T:** Yes. Can we show the same number with fewer checkers?

S: 14 + 6 = 20.



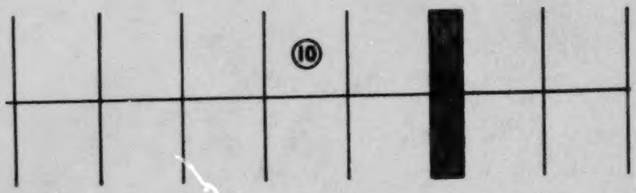
T: Can we make another play now?

S: 70 + 30 = 100.



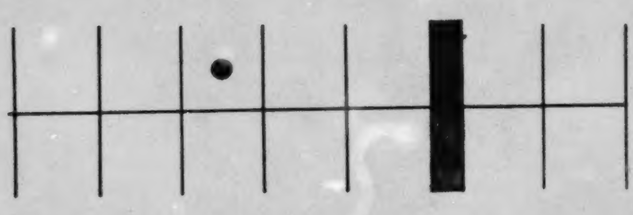
Exercise 2

Display this configuration on the abacus.



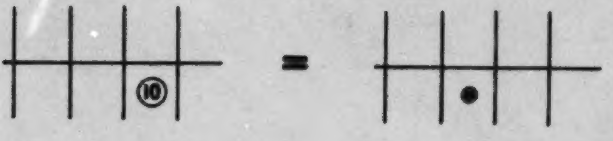
T: Who can show us the same number using a regular checker rather than this ⑩-checker?

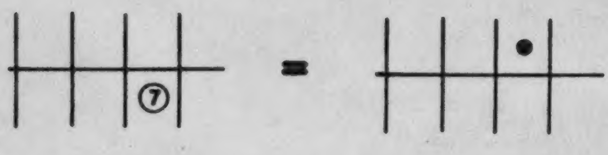
Call on a student to make the trade.

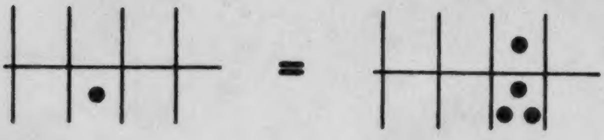


# N 10

In a similar manner lead the class to observe these rules of the 7-abacus.

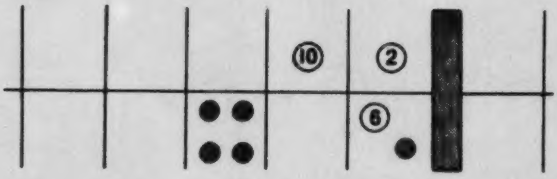






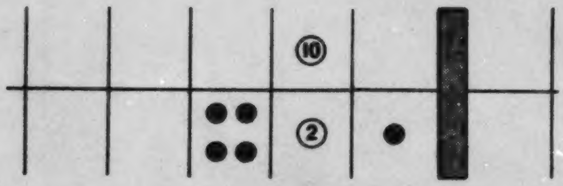
Stress the "forward" and "backward" nature of each rule when a trade is made.

Display this configuration of checkers.



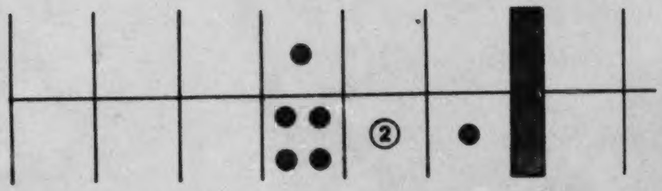
T: Can we show the same number using fewer checkers?

S: 14 + 6 = 20.



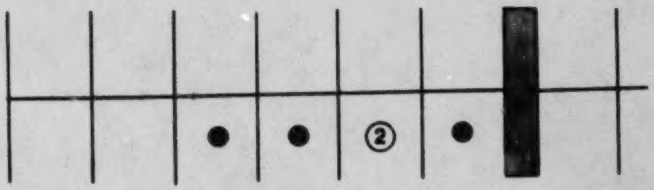
T: Good. Can we make another play and show the same number with still fewer checkers?

S: 10 × 70 = 700.



T: Yes. Can we make another play now?

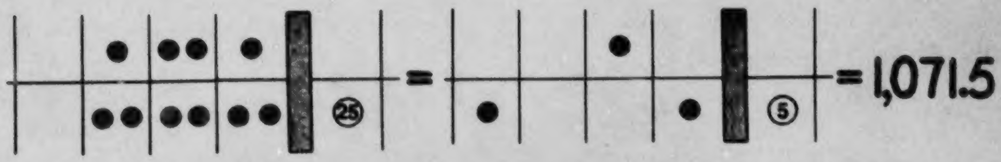
S: 700 + 300 = 1,000.



T: Which number is on the abacus?

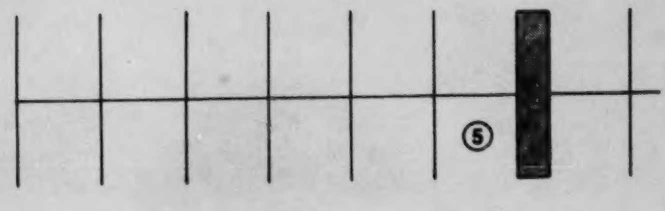
S: 1,121.

Repeat this activity using the configuration of checkers below. Encourage students to announce plays as they make them.



Exercise 3

Put 5 on the 7-abacus.

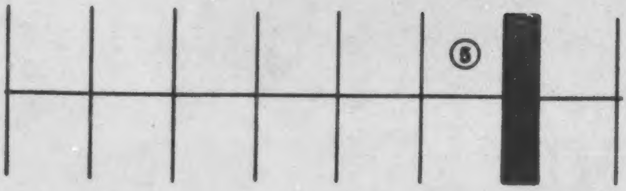


# N 10

**T:** Which number is this?

**S:** 5.

Move the ⑤-checker to the 7-square.



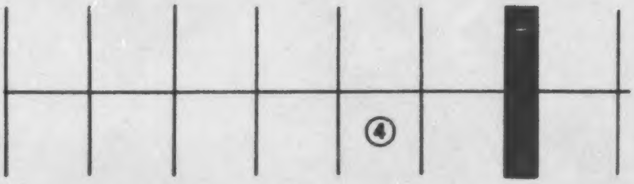
**T:** What number is this?

**S:** 35.

**T:** How many times 5 is 35?

**S:** 7 x 5 = 35.

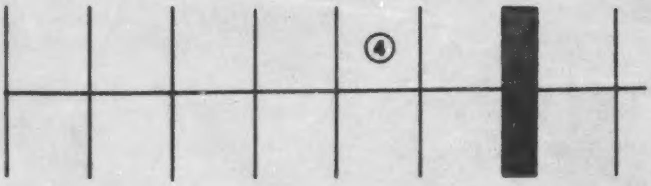
Write this number sentence on the board and clear the abacus. Put a ④-checker on the 10-square.



**T:** What number is this?

**S:** 40, because 4 x 10 = 40.

Move the ④ -checker to the square directly above



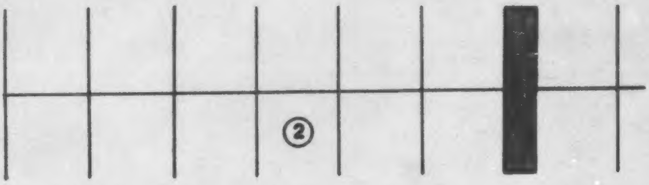
**T:** What number is this?

**S:** 280.

**T:** How many times 40 is 280?

**S:** 7 x 40 = 280.

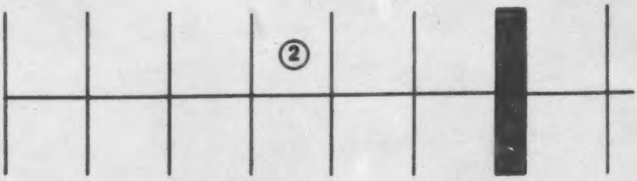
Write this number sentence on the board and clear the abacus. Continue this activity by putting a ② -checker on the 100-square.



**T:** What number is this?

**S:** 200.

Move the ② -checker to the square directly above.



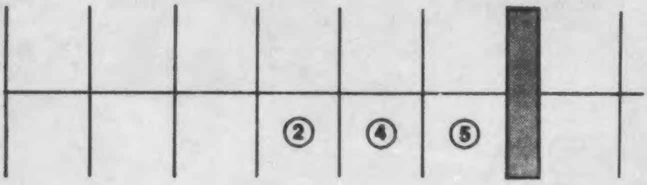
**T:** And what number is this?

**S:** 1,400.

**T:** How many times 200 is 1,400?

**S:** 7 x 200 = 1,400.

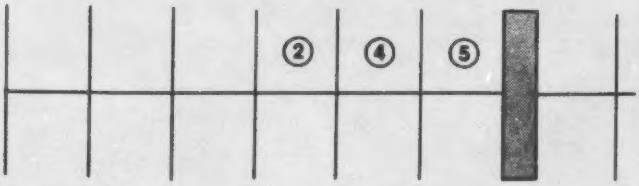
Write this number sentence on the board and then put this configuration on the abacus.



**T:** What number is this?

**S:** 245.

**T:** Now I will move each checker to the square above. What can you tell me about the new number?



Give the class a few minutes to consider this situation before asking for a response.

**S:** The number is 7 x 245.

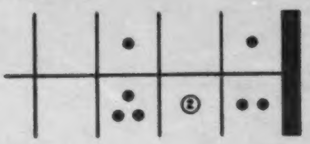
**S:** 1,400 + 280 + 35 = 1,715.

Worksheets N55 and N56 are available.

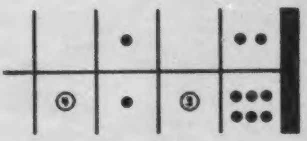
Name

N 55 *

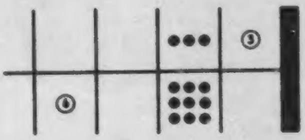
Decode these numbers on Nabu's 7-abacus.



= 1,029



= 4,850



= 6,321

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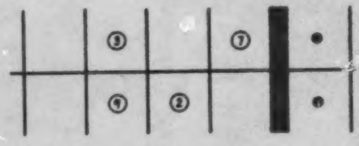
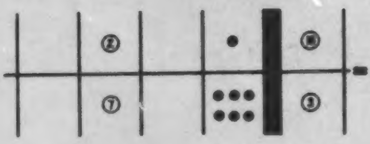
Name

N 56 **

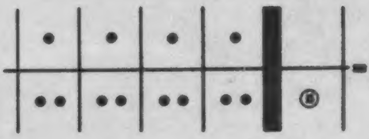
Decode these numbers on Nabu's 7-abacus.



= 3,069.8



= 2,1203



= 10,000.5

## 6.2 A 7-Abacus #2

CAPSULE LESSON SUMMARY

Review the rules of the new 7-abacus introduced in Lesson N6.1 and practice decoding numbers and making plays on it. Investigate the effect of moving or trading certain checkers in a configuration on the abacus. Change a number on the abacus by a specified amount moving exactly one checker. Examine special configurations for numbers which allow the abacus to provide an algorithm for dividing by 7.

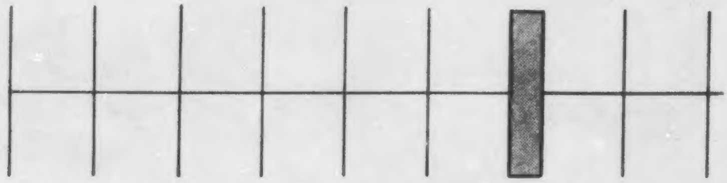
MATERIALS

Teacher : Minicomputer checkers (optional)
Student : Paper; pencil; Worksheets N57 and N58

DESCRIPTION OF LESSON

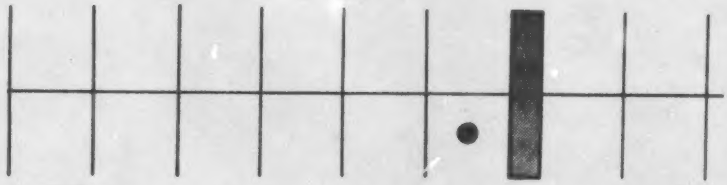
Exercise 1

Draw this abacus on the board.



T: Do you remember Nabu's 7-abacus.

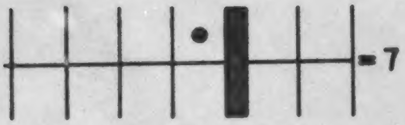
Put (draw) a checker on the 1-square as shown in the illustration below.

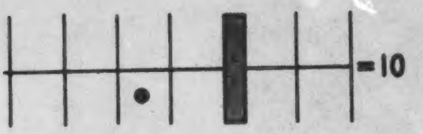


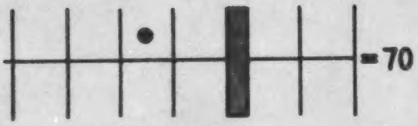
T: Which number is this?

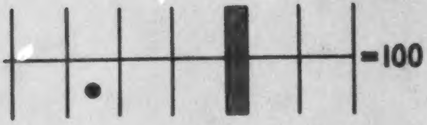
S: 1.

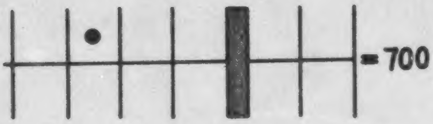
Continue reviewing the values of the squares by moving the checker and asking for the number.

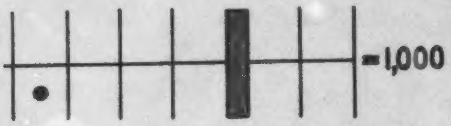


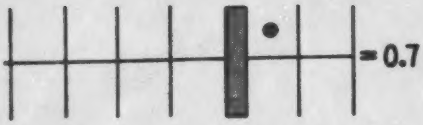


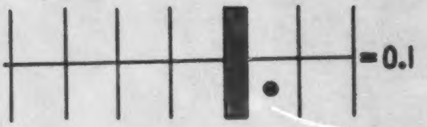


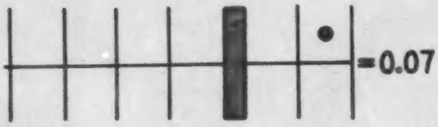


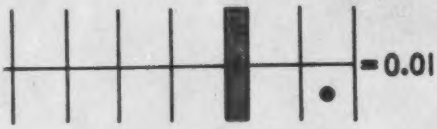






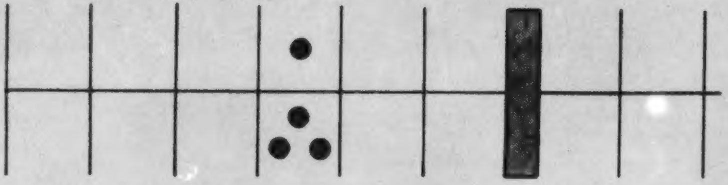






## N 16

Display this configuration of checkers.

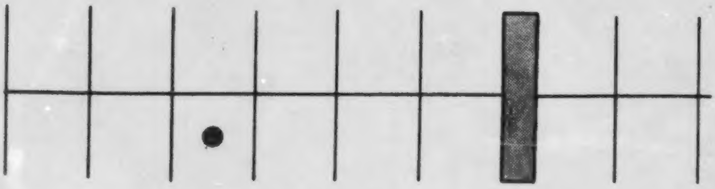


**T:** Which number is this?

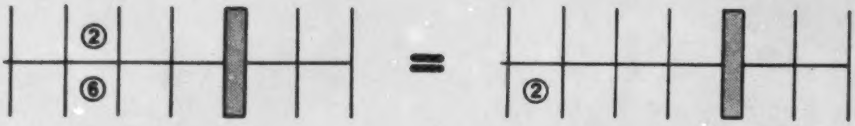
**S:** 1,000.

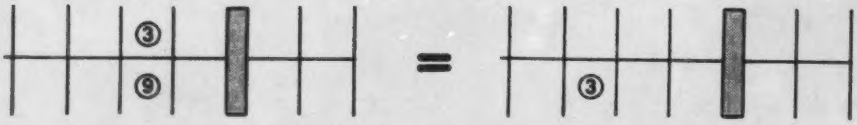
**T:** Can we show 1,000 using exactly one checker?

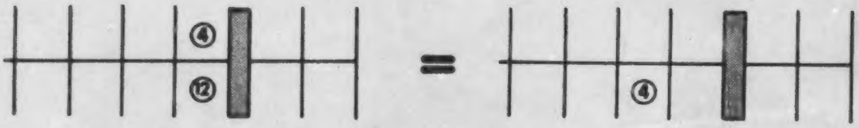
**S:** Yes, 700 + 300 = 1,000.



Continue this activity by asking the class to make some other plays. Several suggestions are given below. Emphasize the "forward" and "backward" nature of these plays.

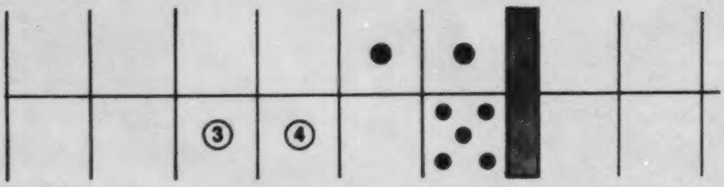






Exercise 2

Display this configuration of checkers.



T: Write this number on your paper.

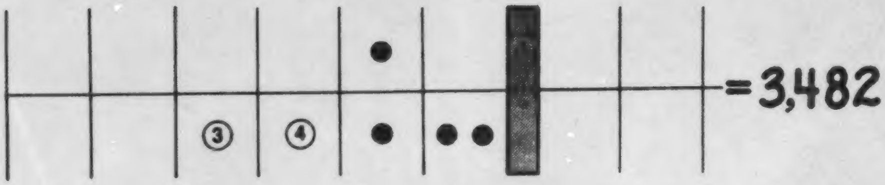
Allow a minute or two for independent work.

S: 3,482.

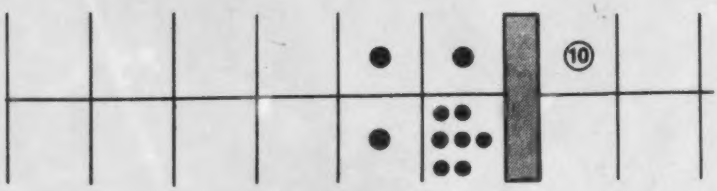
T: Yes. Can we show the same number with fewer checkers?

Call on a student to make a play.

S: 7 + 3 = 10.



Put this configuration of checkers on the abacus.



# N 16

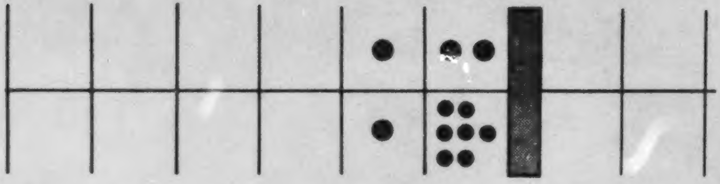
**T:** Write this number on your paper.

Allow a few minutes for independent work.

**S:** 101.

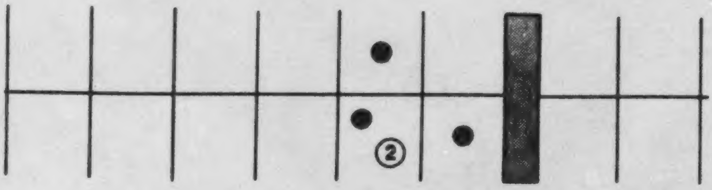
**T:** Good. Can we show the same number with fewer checkers?

**S:** 10 × 0.7 = 7.

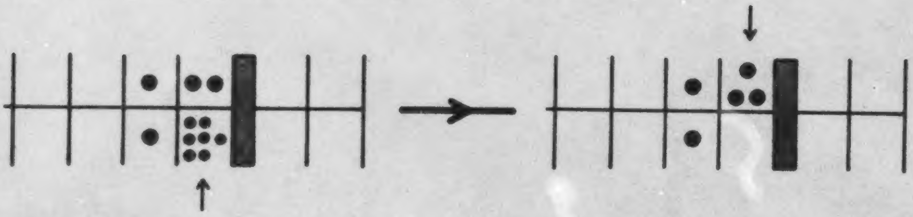


**T:** Can we make other plays to show this number with still fewer checkers?

**S:** 14 + 6 = 20.



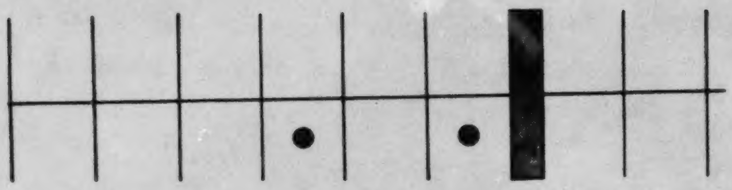
NOTE: It is possible that this play might be made:



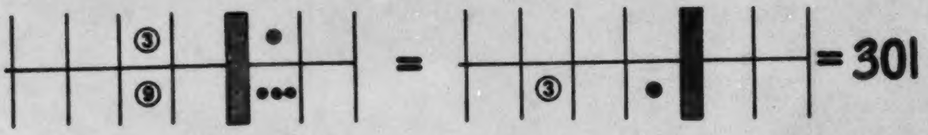
It is correct; however, if it is not made, do not point it out.

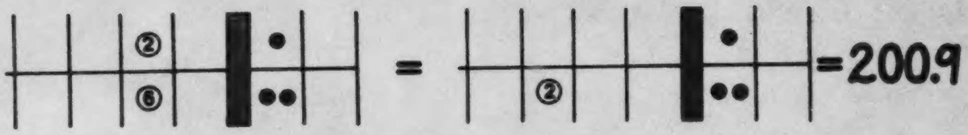
**T:** Good. There is another play we can make.

S: 70 + 30 = 100.



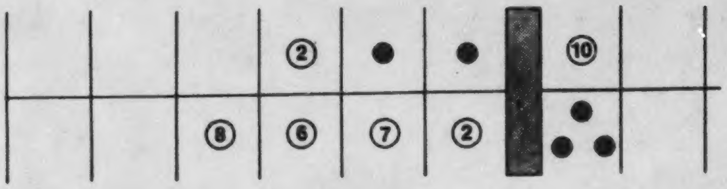
Ask the students to decode two more numbers, and then to make plays. Some numbers that might be used are given below.





Exercise 3

Put this configuration of checkers on the abacus.



T: Look carefully at the number I put on the abacus. What is your estimate of this number? Is it larger than 5,000?

S: Yes, it is larger than 10,000 because there is 8,000 on the thousands' board and 2,000 on the hundreds' board.

# N 16

T: Very good. Now, I will do something either moving some checkers or trading some checkers. Each time you must tell me if we get a larger number, a smaller number, or the same number.

Trade the ⑩-checker on the 0.7-square for a regular checker on the 7-square.

S: The number is the same, because 10 x 0.7 = 7.

Move a regular checker from the 7-square to the 70-square.

S: The number is larger.

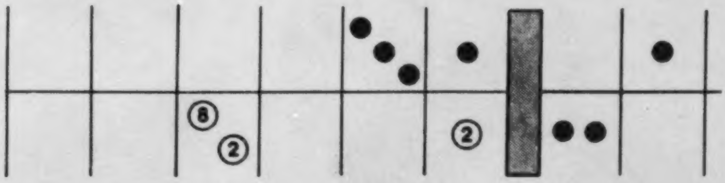
T: How much larger?

S: 63 larger, because 7 + 63 = 70.

Continue in this manner making the following moves.

- Trade the ⑦ -checker on the 10-square for a regular checker on the 70-square. (Same)
- Move a regular checker on the 0.1-square to the 0.07-square. (0.03 smaller; 0.1 - 0.07 = 0.03)
- Trade the ② -checker on the 700-square and the ⑥ -checker on the 100-square for a ②-checker on the 1,000-square. (Same)

The configuration on the abacus should be the one below, at this point.



**T:** How can we make this number 3 larger by moving exactly one checker from one square to another square?

**S:** Move a regular checker from the 7-square to the 10-square.

**T:** How can we make this number 1,998 smaller by moving exactly one checker?

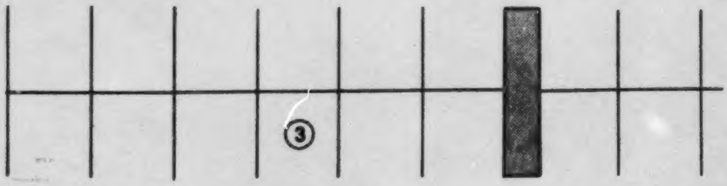
**S:** Move the ②-checker on the 1,000-square to the 1-square; 2,000 - 2 = 1,998.

Continue this activity by asking the class to:

- Increase the number by 630.
  (Move a regular checker from the 70-square to the 700-square; 700 - 70 = 630.)
- Decrease the number by 7,200.
  (Move the ⑧-checker from the 1,000-square to the 100-square; 8,000 - 800 = 7,200.)
- Increase the number by 198.
  (Move a ②-checker from the 1-square to the 100-square; 200 - 2 = 198.)
- Decrease the number by 0.09.
  (Move a regular checker from the 0.1-square to the 0.01-square; 0.1 - 0.01 = 0.09.)

Exercise 4

Clear the abacus and put on 300 with this configuration.

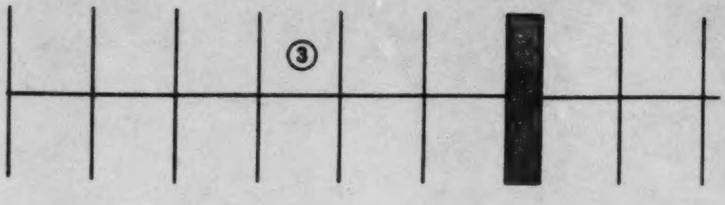


# N 16

T: Which number is this?

S: 300.

Move the ③-checker to the 700-square.



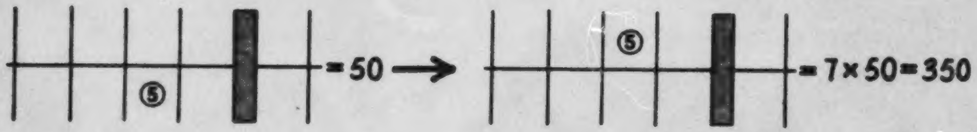
T: What number is this?

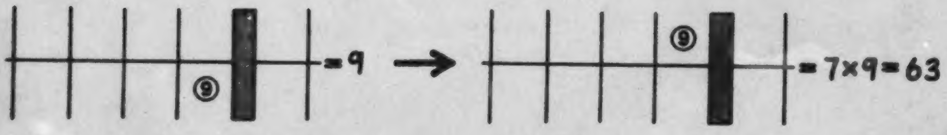
S: 2,100.

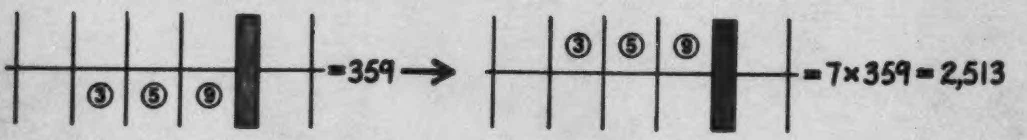
T: How many times 300 is 2,100?

S: 7 x 300 = 2,100.

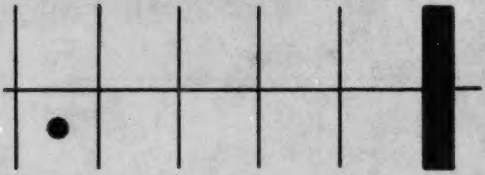
Write this number sentence on the board. Continue in this manner making the moves suggested in the sequence below. Stress that when checkers are moved from bottom squares to squares directly above, the new number is seven times the original number.







Clear the abacus and put on 10,000 with one regular checker.

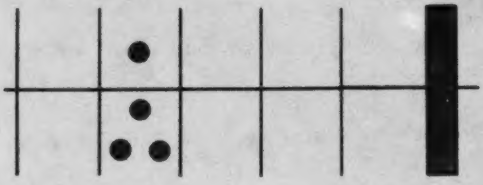


T: Which number is this?

S: 10,000.

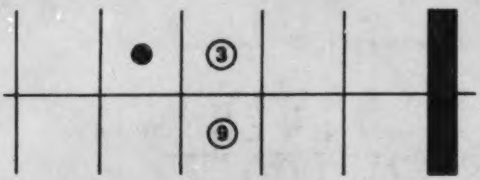
T: Let's try to show 10,000 another way with checkers on the upper squares as much as possible. Who can make a move?

S: 10,000 = 7,000 + 3,000.



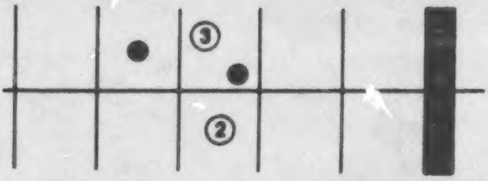
T: Who can make another play?

S: 3,000 = 2,100 + 900.

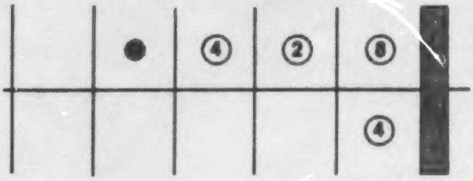


# N 16

S: 900 = 700 + 200.



Continue to call on students to make plays. Do not be concerned if nonstandard plays are made. After several plays, the abacus should look like this. Remember, the object is to get checkers on the upper squares as much as possible.



T: The abacus shows 10,000 in a special way. On your paper, write a name for 10,000 as suggested by this configuration on the 7-abacus.

Allow the students to work independently for a few minutes.

S: 10,000 = (7 x 1,428) + 4.

Write the above number story on the board.

T: Calculate "10,000 ÷ 7".

Ask a student to write a solution to this calculation on the board.

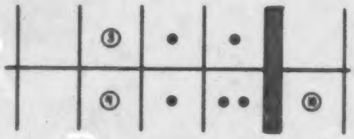
$$\begin{array}{r} 1{,}428 \text{ R} = 4 \\ 7\overline{)10{,}000} \end{array}$$

Worksheets N57 and N58 are available.

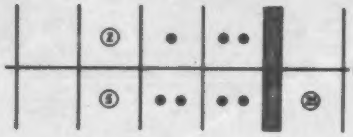
Name ______________________

N 57 *

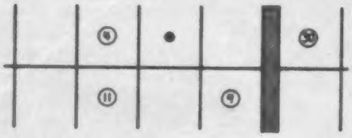
Decode these numbers on Nabu's 7-abacus.



= 5,090



= 2,008.5



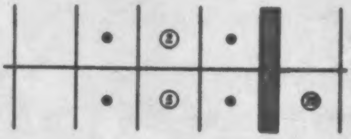
= 4,000

---

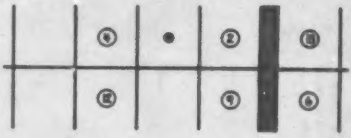
Name ______________________

N 58 **

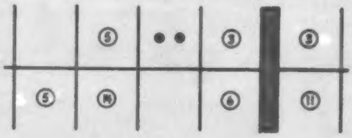
Decode these numbers on Nabu's 7-abacus.



= 1,000



= 4,100.6



= 10,075.2

# N 19

## 6.3 Division Algorithm #1

CAPSULE LESSON SUMMARY

Solve division problems suggested by situations in which a given amount of money is shared equally among some specified number of people. Present a form of the standard division algorithm with decimals.

MATERIALS

Teacher: None
Student: Paper; pencil; Worksheets N59 and N60

DESCRIPTION OF LESSON

Begin the lesson with a brief mental arithmetic activity using multiplication problems such as the ones suggested here. Feel free to adjust the problems to suit the numerical abilities of your students. Answers are in the boxes.

$25 \times 10 = \boxed{250}$
$2.5 \times 10 = \boxed{25}$
$0.25 \times 10 = \boxed{2.5}$

$32 \times 40 = \boxed{1{,}280}$
$3.2 \times 40 = \boxed{128}$
$0.32 \times 40 = \boxed{12.8}$

$43 \times 70 = \boxed{3{,}010}$
$4.3 \times 70 = \boxed{301}$
$0.43 \times 70 = \boxed{30.1}$

Exercise 1

Draw this table on the board.

| Amount of Money | Number of People | Amount of Money for Each Person | Amount of Money Left Over |
|---|---|---|---|
| $20.80 | 4 | | |

Point to the appropriate columns of the table as you describe the problem.

T: Four people want to share $20.80 as equally as possible. How much money will each person receive and how much money will be left over, if any?

S: Each person will receive $5.00 and $0.80 will be left over.

T: Could the four people share $0.80 equally?

S: Yes, each person would receive $.20.

S: So altogether each person will receive $5.20, because 4 × 5.20 = 20.80.

Add this information to your table.

| Amount of Money | Number of People | Amount of Money for Each Person | Amount of Money Left Over |
|---|---|---|---|
| $20.80 | 4 | $5.20 | 0 |

Continue this activity giving the information in the first two columns of the table below and asking students to provide the numbers in the boxes. Feel free to adjust the level of difficulty to the numerical abilities of your class.

| Amount of Money | Number of People | Amount of Money for Each Person | Amount of Money Left Over |
|---|---|---|---|
| $20.80 | 4 | $5.20 | 0 |
| $110.76 | 10 | $11.07 | $0.06 |
| $36.96 | 6 | $6.16 | 0 |
| $45.60 | 9 | $5.06 | $0.06 |
| $67.20 | 8 | $8.40 | 0 |
| $57.20 | 8 | $7.15 | 0 |
| $78.21 | 11 | $7.11 | 0 |

# N 19

**Exercise 2**

T: Fifteen people wish to share equally $1,555.37. How much money would each person receive? Would each person receive as much as $100?

S: Yes, 15 x 100 = 1,500.

T: Would each person receive as much as $200?

S: No; 15 x 200 = 3,000.

T: So each person would receive between $100 and $200. Let's first give each person $100.

Record this information by writing a calculation on the board.

$$\begin{array}{rr} 1{,}555.37 & \\ -\ 1{,}500.00 & 100 \\ \hline 55.37 & \end{array}$$

T: Now, we must share $55.37 among the fifteen people. What do you suggest?

S: 15 x 3 = 45 and 15 x 4 = 60, so give each person $3.00 more.

Add this information to the calculation on the board.

$$\begin{array}{rr} 1{,}555.37 & \\ -\ 1{,}500.00 & 100 \\ \hline 55.37 & \\ -\ 45.00 & 3 \\ \hline 10.37 & \end{array}$$

T: How much money remains to be shared?

S: $10.37.

T: What do you suggest we do?

S: 15 × 0.6 = 9.00 and 15 × 0.7 = 10.50, so give each person $0.60 more.

Record this information on the board.

$$\begin{array}{rr} 1{,}555.37 & \\ -\,1{,}500.00 & 100 \\ \hline 55.37 & \\ -\,45.00 & 3 \\ \hline 10.37 & \\ -\,9.00 & 0.60 \\ \hline 1.37 & \end{array}$$

S: 15 × 0.09 = 1.35, so give each person $0.09 more.

Add this information to the calculation on the board.

$$\begin{array}{rr} 1{,}555.37 & \\ -\,1{,}500.00 & 100 \\ \hline 55.37 & \\ -\,45.00 & 3 \\ \hline 10.37 & \\ -\,9.00 & 0.60 \\ \hline 1.37 & \\ -\,1.35 & 0.09 \\ \hline 0.02 & \end{array}$$

T: How much money is left over?

S: Two cents.

T: What should we do?

Let the students discuss this situation for a few minutes. Some students will probably suggest that, practically, the two cents cannot be shared among the fifteen people and that the problem is finished. Other students may suggest that the process could continue by thinking about fractions of cents. If this later idea is not suggested, it is not necessary for you to mention it at this time.

T: How much money does each person receive?

S: Each person receives $103.69 and $0.02 is left over.

T: Yes. This problem of sharing $1,555.37 among fifteen people is a division problem. I'll show you a way to write the calculation.

Amend the calculation on the board to indicate that it is the calculation for a division problem.

```
         103.69    R=0.02
      ___________
   15)1,555.37  |
     -1,500.00  |   100
  --------------|---------
         55.37  |
        -45.00  |     3
  --------------|---------
         10.37  |
         -9.00  |     0.60
  --------------|-----------
          1.37  |
         -1.35  |     0.09
  --------------|-----------
          0.02  |   103.69
```

- Repeat this activity by solving another division problem such as the one below, either collectively or independently.

$$6\overline{)157.32}$$

If you choose to ask the students to work independently, check the problem collectively. The solution is given below.

$$\begin{array}{r|r} 26.22 & \\ 6\overline{)157.32} & \\ -156.00 & 26 \\ \hline 1.32 & \\ -1.20 & 0.20 \\ \hline 0.12 & \\ 0.12 & 0.02 \\ \hline 0 & 26.22 \end{array}$$

NOTE: When the remainder is 0, it need not be recorded in the solution to a calculation.

Write these one-star division problems on the board and instruct the students to solve them independently. Solutions are in the boxes.

$$\begin{array}{r} \boxed{22.10} \\ 5\overline{)110.50} \end{array} \qquad \begin{array}{r} \boxed{120.04 \text{ R-}0.04} \\ 8\overline{)960.36} \end{array}$$

$$\begin{array}{r} \boxed{49.49} \\ 25\overline{)1{,}237.25} \end{array} \qquad \begin{array}{r} \boxed{117.30 \text{ R=}0.19} \\ 20\overline{)2{,}346.19} \end{array}$$

# N 19

After a short while, check the problems collectively by asking students to explain the calculations at the board.

Worksheets N59 and N60 are available for additional independent work.

Name ______________________ N 59 **

Complete each problem.

```
       27.24  R=0.01
   6 ) 163.45
     - 120.00 | 20
        43.45 |
      - 42.00 |  7
         1.45 |
       - 1.20 |  0.20
         0.25 |
       - 0.24 |  0.04
         0.01 | 27.24
```

---

```
        17.98  R=0.62
   67 ) 1205.28
       - 670.00 | 10
         535.28 |
       - 469.00 |  7
          66.28 |
         -60.30 |  0.9
           5.98 |
          -5.36 |  0.08
           0.62 | 17.98
```

---

Name ______________________ N 60 ***

Complete each problem.

```
        35.04  R=0.02
   15 ) 525.62
      - 450.00 | 30
         75.62 |
       - 75.00 |  5
          0.62 |
        - 0.60 |  0.04
          0.02 | 35.04
```

---

```
          28.6  R=1.06
   125 ) 3576.06
         -2500.00 | 20
          1076.06 |
         -1000.00 |  8
            76.06 |
           -75.00 |  0.6
             1.06 | 28.6
```

352

# N 23

## 6.4 A 7-Abacus #3

CAPSULE LESSON SUMMARY

Review the rules and plays of the 7-abacus introduced in Lesson N6.1. Practice decoding numbers on the abacus and using plays to make the numbers easier to read. Investigate the effect of moving or trading certain checkers in a configuration on the abacus. Change a number on the abacus by a specified amount moving exactly one checker. Examine special configurations for numbers which allow the abacus to provide an algorithm for dividing by 7.

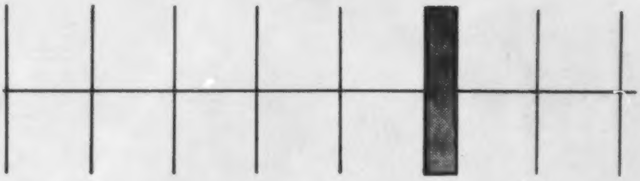
MATERIALS

| | |
|---|---|
| Teacher: | Minicomputer checkers (optional) |
| Student: | Paper; pencil; Worksheets N61 and N62 |

### DESCRIPTION OF LESSON

#### Exercise 1

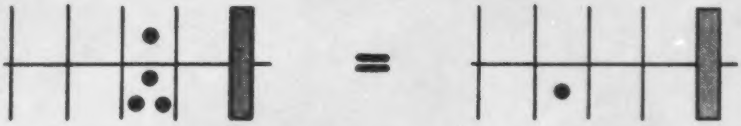
Draw this abacus on the board.



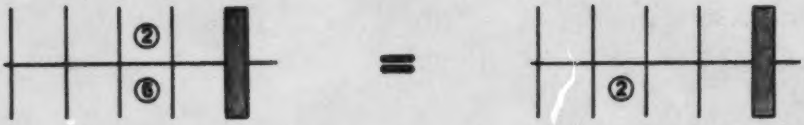
Review the values of the squares on the 7-abacus by moving a checker quickly from one square to another: 1; 7; 10; 70; 100; 700; 1,000; 7,000; 0.1; 0.7; 0.01; 0.07.

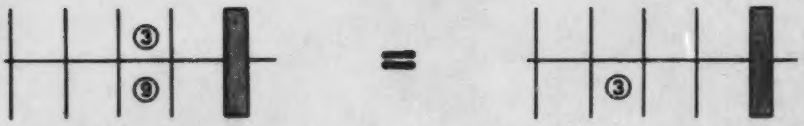
Review trades that can be made on the abacus emphasizing that:

> One checker on an upper square and three checkers on the square directly below may be traded for one checker on the next lower square to the left.

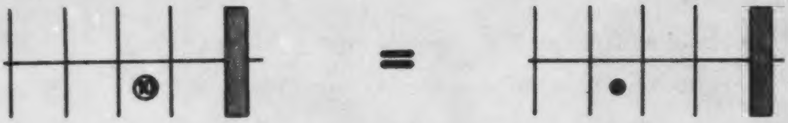


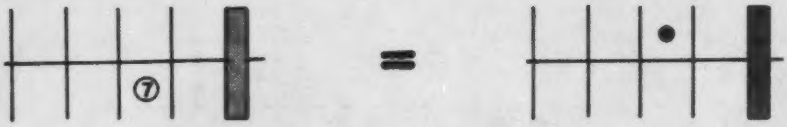
Direct the class to observe other trades that result from this basic trade. For example,



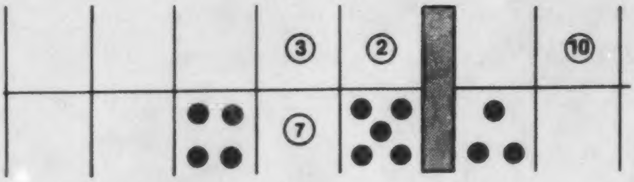


Also review these trades.





Display this configuration of checkers.



T: Write this number on your paper.

Let the students work independently for a few minutes.

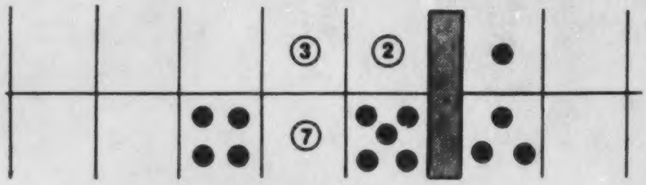
S: 700.

# N 23

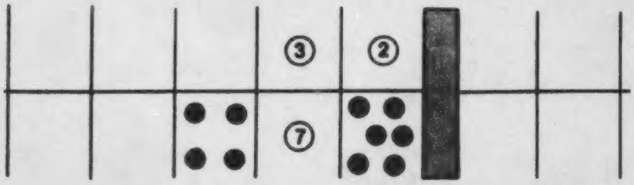
**T:** Let's check by making some plays to show the same number with fewer checkers.

Call on students to make plays.

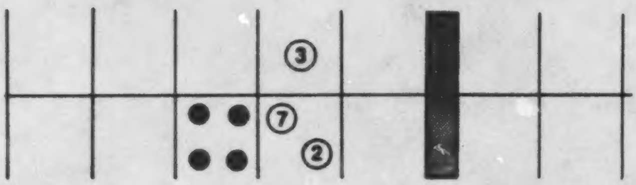
**S:** Ten checkers on the 0.07-square is the same as one checker on the 0.7-square.



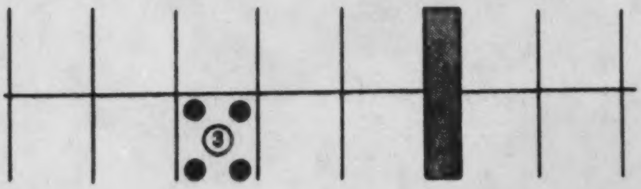
**S:** One checker (on the 0.7-square) and three checkers (on the 0.1-square) trades for one checker (on the 1-square) because 0.7 + 0.3 = 1.



**S:** Two checkers (on the 7-square) and six checkers (on the 1-square) trades for two checkers on the 10-square. (14 + 6 = 20).

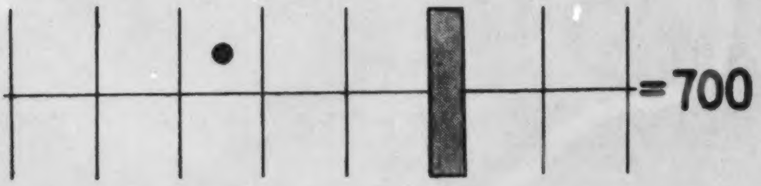


S: Three checkers (on the 70-square) and nine checkers (on the 10-square) trades for three checkers on the 100-square.

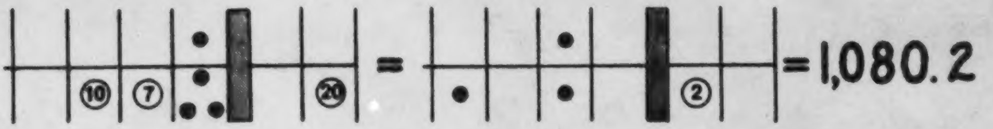


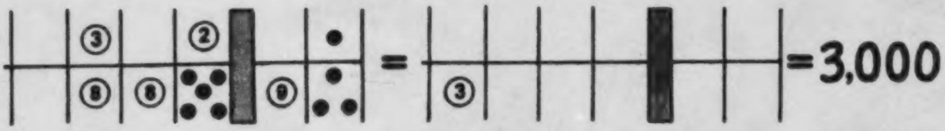
T: Can someone make another play?

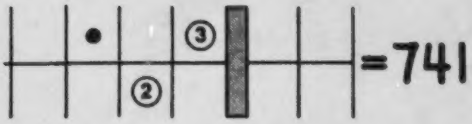
S: Seven checkers on the 100-square is the same as one checker on the 700-square.



Continue the decoding using the configurations suggested below or others that are more appropriate for the numerical level of your class. Be sure to ask students to make plays where appropriate.



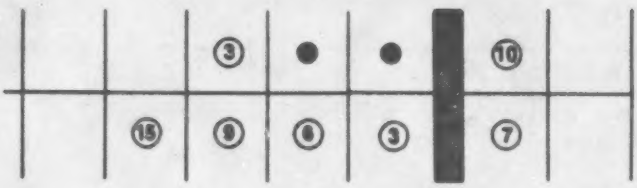




# N 23

## Exercise 2

Display this configuration of checkers.



T: Look carefully at the number I put on the abacus. What is your estimate of this number?

If an estimate is not forthcoming, lead the class by asking if the number is larger than 15,000 (yes); larger than 18,000.

S: It is larger than 18,000 because there is 15,000 on the thousands' board and 3,000 on the hundreds' board.

T: I will do something either moving some checkers or trading some checkers. Each time you are to tell me if we get a larger number, a smaller number, or the same number as before.

Move a regular checker from the 7-square to the 70-square.

S: The number is larger.

T: How much larger . . . ?

S: 63 larger, because 7 + 63 = 70.

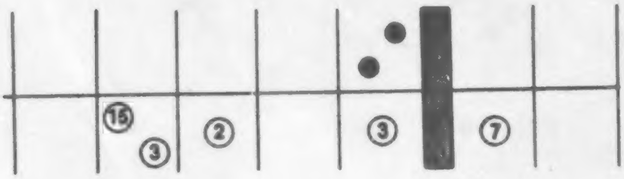
Trade a ⑩ -checker on the 0.7-square for a regular checker on the 7-square.

S: The number is the same, because 10 × 0.7 = 7.

Continue in this manner making the following moves:

- Trade a ⑥-checker on the 10-square for a regular checker on the 7-square.
  (53 smaller; 60 - 53 = 7)
- Trade a ③-checker on the 700-square and a ⑨-checker on the 100-square for a ③-checker on the 1,000-square.
  (Same; 2,100 + 900 = 3,000)
- Trade two regular checkers on the 70-square for a ②-checker on the 100-square.
  (60 larger; 140 + 60 = 200)

At this point, the configuration on the abacus should be the one below.



T: Who can make this number 27 larger by moving exactly one checker from one square to another square?

S: Move the ③-checker from the 1-square to the 10-square.

T: Who can make this number 198 smaller by moving exactly one checker?

S: Move the ②-checker from the 100-square to the 1-square.

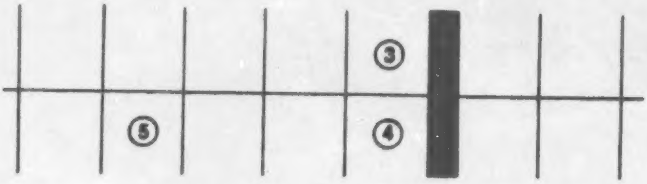
Continue this activity by asking the class to:

- Increase the number by 693.
  (Move a regular checker from the 7-square to the 700-square; 7 + 693 = 700.)

- Decrease the number by 0.6.
  (Move a ②-checker from the 1-square to the 0.7-square; 2 - 0.6 = 1.4.)

**Exercise 3**

Clear the abacus and put on this configuration.



**T:** Again, I will do something either moving some checkers or trading some checkers. Tell me if the new number is larger, smaller, or the same as the previous number.

Move a ③-checker from the 7-square to the 70-square.

**S:** The new number is 189 larger.

**T:** How did you get 189?

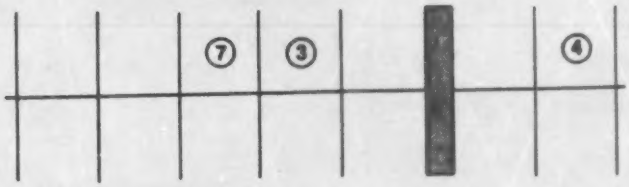
**S:** 21 + 189 = 210.

Continue this activity making the moves or trades suggested below.

- Trade a ⑤-checker on the 1,000-square for a ⑤-checker on the 700-square and a ⑮-checker on the 100-square.
  (Same; 5,000 = 3,500 + 1,500)
- Move a ④-checker from the 1-square to the 0.1-square.
  (3.6 smaller; 4 - 0.4 = 3.6)

- Trade a ⑮ -checker on the 100-square for a ② -checker on the 700-square.
  (100 smaller; 1,500 - 1,400 = 100)
- Move a ④ -checker from the 0.1-square to the 0.07-square.
  (0.12 smaller; 0.4 - 0.12 = 0.28)
- Trade a ⑤ -checker and a ② -checker on the 700-square for a ⑦ -checker on the 700-square.
  (Same; 5 + 2 = 7)

At this point, the configuration on the abacus should be:



T: Look at this number carefully. Write this number on your paper. Try to find a name for the number suggested by the abacus.

S: The number is 7 x 730.04.

T: Yes. Calculate "7 x 730.04".

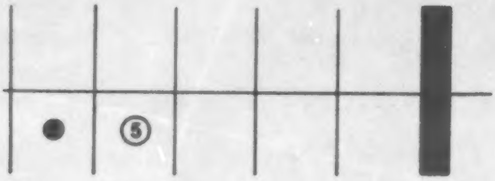
Give the students a few minutes to do the calculation on their papers. Then write a number story on the board as a student announces the result.

S: 7 x 730.04 = 5,110.28.

# N 23

**Exercise 4**

Clear the abacus and put on this configuration.

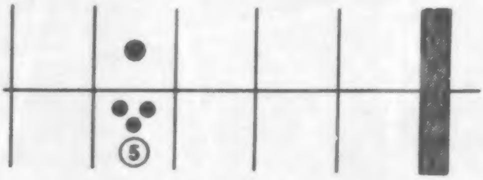


T: Which number is this?

S: 15,000.

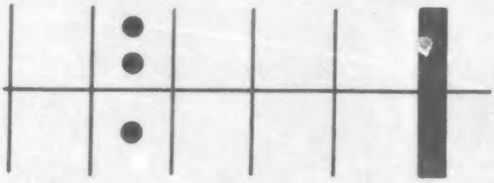
T: Let's try to show 15,000 with checkers on the upper squares as much as possible. Who can make a play?

S: One checker on the 10,000 square is the same as one checker on the 7,000-square and three checkers on the 10,000-square.

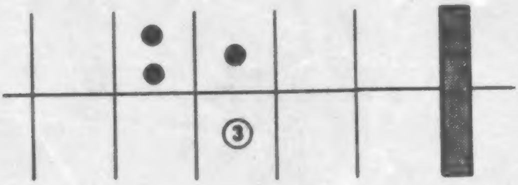


T: Who can make another play?

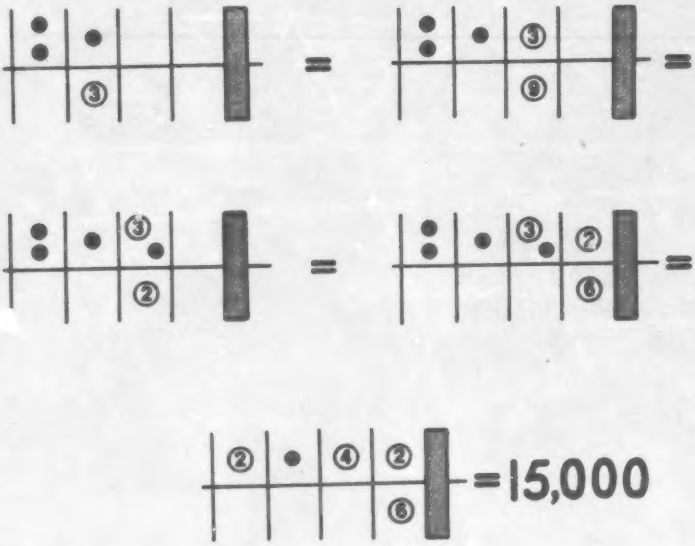
S: 8,000 = 7,000 + 1,000.



S: One checker on the 1,000-square is the same as one checker on the 700-square and three checkers on the 100-square.



Continue to call on students to make plays. Do not be concerned if non-standard plays are made as long as the number is not changed. A possible sequence of plays is given below.



T: The abacus shows 15,000 in a special way. On your paper, write another name for 15,000 as suggested by the abacus.

Allow a few minutes for students to work independently. Then write a number story on the board as a student announces:

S: 15,000 = (7 x 2,142) + 6.

# N 23

T: Calculate "15,000 ÷ 7".

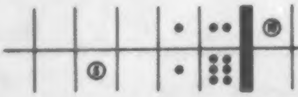
Ask a student to write the problem and solution on the board.

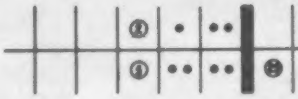
$$\begin{array}{l} \quad\; 2,142 \;\; R=6 \\ 7\overline{)15,000} \end{array}$$

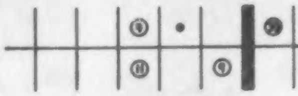
Worksheets N61 and N62 are available.

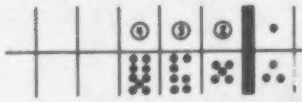
Name ______________________ N 61 *

Decode these numbers on Nabu's 7-abacus.

 = 5,107

 = 2,008.5

 = 4,000

 = 4,000

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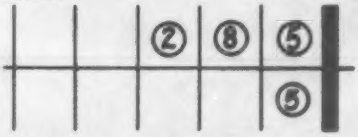
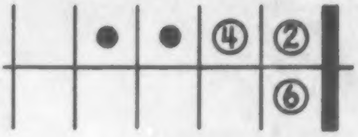
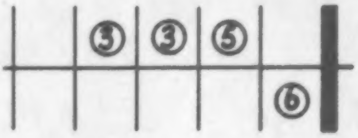
Name ______________________ N 62 **

Write each number on Nabu's 7-abacus by putting as many checkers as you can in the upper squares.

2,000 = 

8,000 = 

23,456 = 

# N 27

## 6.5 Division Algorithm #2

CAPSULE LESSON SUMMARY

Solve division problems suggested by situations in which a given amount of money is shared equally among some specified number of people. Present a form of the standard division algorithm with decimals.

MATERIALS

Teacher: None
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Begin the lesson with a brief mental arithmetic activity using division problems such as the ones suggested here. Feel free to adjust the problems to suit the numerical abilities of your students. Answers are in the boxes.

$$2{,}000 \div 10 = \boxed{200} \qquad 3{,}560 \div 10 = \boxed{356}$$

$$2{,}000 \div 100 = \boxed{20} \qquad 3{,}560 \div 100 = \boxed{35.6}$$

$$2{,}000 \div 1{,}000 = \boxed{2} \qquad 3{,}560 \div 1{,}000 = \boxed{3.56}$$

$$4{,}680 \div 20 = \boxed{234}$$

$$4{,}680 \div 200 = \boxed{23.4}$$

$$4{,}680 \div 2{,}000 = \boxed{2.34}$$

Exercise 1

Draw this table on the board.

| Amount of Money | Number of People | Amount of Money for Each Person | Amount of Money Left Over |
|---|---|---|---|
| $54.72 | 6 | | |

Point to the appropriate columns of the table as you describe the problem.

T: Six people want to share $54.72 as equally as possible. How much money will each person receive and how much money will be left over, if any?

Allow a few minutes for students to work on this problem.

S: Each person will receive $9.12, because 6 x 9.12 = 54.72.

Add this information to your table.

| Amount of Money | Number of People | Amount of Money for Each Person | Amount of Money Left Over |
|---|---|---|---|
| $54.72 | 6 | $9.12 | 0 |

Continue this activity giving the information in the first two columns of the table below and asking students to provide the numbers in the boxes. Feel free to adjust the level of difficulty to the numerical abilities of your class.

| Amount of Money | Number of People | Amount of Money for Each Person | Amount of Money Left Over |
|---|---|---|---|
| $54.72 | 6 | $9.12 | 0 |
| $35.79 | 5 | $7.15 | $0.04 |
| $75.20 | 8 | $9.40 | 0 |
| $60.30 | 4 | $15.07 | $0.02 |
| $5.04 | 4 | $1.26 | 0 |
| $65.34 | 4 | $16.33 | $0.02 |

Exercise 2

T: Twelve people wish to share equally $1,482.55. How much money would each person receive? Would each person receive as much as $100?

S: Yes; 12 x 100 = 1,200.

T: Would each person receive as much as $200?

S: No; 12 x 200 = 2,400.

T: So each person would receive between $100 and $200. Let's first give each person $100.

Record this information by writing a calculation on the board.

$$\begin{array}{rr}
1{,}482.55 & \\
-\,1{,}200.00 & 100 \\
\hline
282.55 &
\end{array}$$

T: Now we must share $282.55 among the twelve people. What do you suggest?

S: 12 x 20 = 240 and 12 x 30 = 360, so give each person $20 more.

Add this information to the calculation on the board.

$$\begin{array}{rr}
1{,}482.55 & \\
-\,1{,}200.00 & 100 \\
\hline
282.55 & \\
-\,240.00 & 20 \\
\hline
42.55 &
\end{array}$$

T: How much money remains to be shared?

S: $42.55.

T: What do you suggest we do?

S: 12 x 3 = 36 and 12 x 4 = 48, so give each person $3.00 more.

Record this information on the board.

$$
\begin{array}{rr}
1{,}482.55 & \\
-\,1{,}200.00 & 100 \\
\hline
282.55 & \\
-\,240.00 & 20 \\
\hline
42.55 & \\
36.00 & 3 \\
\hline
6.55 &
\end{array}
$$

S: 12 x 0.5 = 6 and 12 x 0.6 = 7.2, so give each person $0.50 more.

Add this information to the calculation on the board.

$$
\begin{array}{rr}
1{,}482.55 & \\
-\,1{,}200.00 & 100 \\
\hline
282.55 & \\
-\,240.00 & 20 \\
\hline
42.55 & \\
-\,36.00 & 3 \\
\hline
6.55 & \\
-\,6.00 & \underline{0.50} \\
\hline
0.55 &
\end{array}
$$

S: 12 x 0.04 = 0.48 and 12 x 0.05 = 0.60, so give each person $0.04 more.

Record this information on the board.

$$
\begin{array}{rr}
1{,}482.55 & \\
-\,1{,}200.00 & 100 \\
\hline
282.55 & \\
-\,240.00 & 20 \\
\hline
42.55 & \\
-\,36.00 & 3 \\
\hline
6.55 & \\
-\,6.00 & 0.50 \\
\hline
0.55 & \\
-\,0.48 & 0.04 \\
\hline
0.07 &
\end{array}
$$

T: How much money is left over?

S: Seven cents.

T: What should we do?

Let the students discuss this situation for a few minutes. Some students will probably suggest that, practically, the seven cents cannot be shared among the twelve people and that the problem is finished. Other students may suggest that the process could continue by thinking about fractions of cents. If this idea is not suggested by anyone, it is not necessary for you to mention it at this time.

T: How much money should each person receive?

S: Each person receives \$123.54 and \$0.07 is left over.

T: Yes. This problem of sharing \$1,482.55 among twelve people is a division problem. I'll show you a way to write the calculation.

Amend the calculation on the board to indicate that it is the calculation for a division problem.

$$\begin{array}{r|r}
\multicolumn{1}{r}{123.54} & \multicolumn{1}{l}{R=0.07} \\ \cline{2-2}
12\overline{)1,482.55} & \\
-1,200.00 & 100 \\ \hline
282.55 & \\
-240.00 & 20 \\ \hline
42.55 & \\
-36.00 & 3 \\ \hline
6.55 & \\
-6.00 & 0.50 \\ \hline
0.55 & \\
-0.48 & 0.04 \\ \hline
0.07 & 123.54
\end{array}$$

Repeat this activity by solving another division problem such as the one below, either collectively or independently.

$$8\overline{)207.15}$$

If you choose to ask the students to work independently, check the problem collectively. The solution is given below.

$$\begin{array}{r|r}
\multicolumn{1}{r}{25.89} & \multicolumn{1}{l}{R=0.03} \\ \cline{2-2}
8\overline{)207.15} & \\
-200.00 & 25 \\ \hline
7.15 & \\
-6.40 & 0.80 \\ \hline
0.75 & \\
-0.72 & 0.09 \\ \hline
0.03 & 25.89
\end{array}$$

## N 27

Write these division problems on the board and instruct the students to solve them independently. Solutions are in the boxes.

$$\begin{array}{r} \boxed{43.20} \\ 3\overline{)129.60} \end{array} \qquad \begin{array}{r} \boxed{62.41 \ R=0.03} \\ 5\overline{)312.08} \end{array}$$

$$\begin{array}{r} \boxed{102.30} \\ 9\overline{)920.70} \end{array} \qquad \begin{array}{r} \boxed{154.02} \\ 22\overline{)3{,}388.44} \end{array}$$

$$\begin{array}{r} \boxed{241.01 \ R=0.26} \\ 41\overline{)9{,}881.67} \end{array}$$

After a short while, check the problems collectively by asking students to explain the calculations at the board.

## 6.6 A 7-Abacus #4

CAPSULE LESSON SUMMARY

Review the rules and plays of the 7-abacus introduced in Lesson N6.1. Practice decoding numbers on the abacus and using plays to make numbers easier to read. Use the abacus to solve some division problems which involve sharing amounts of money equally among seven people. Do the calculation "1 ÷ 7" on the abacus to find that $\frac{1}{7} = 0.\overline{142857}$.

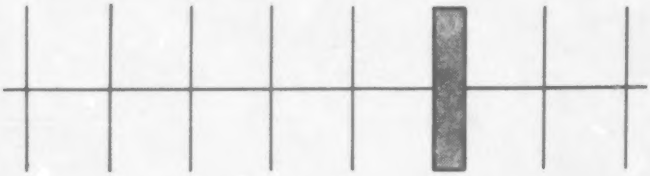
MATERIALS

Teacher: Minicomputer checkers (optional); hand-calculator
Student: Paper; pencil

### DESCRIPTION OF LESSON

Exercise 1

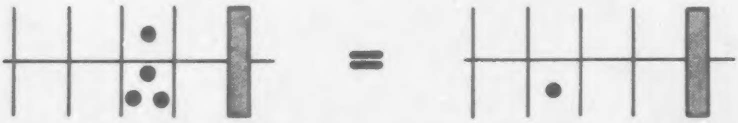
Draw this abacus on the board.



Review the values of the squares on a 7-abacus by moving a checker quickly from one square to another: 1; 7; 10; 70; 100; 700; 1,000; 7,000; 0.1; 0.7; 0.01; 0.07.

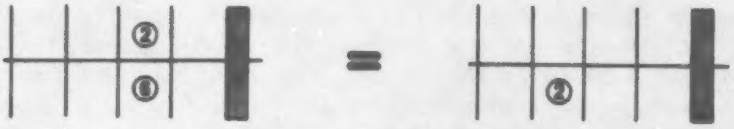
Review trades that can be made on the abacus emphasizing that:

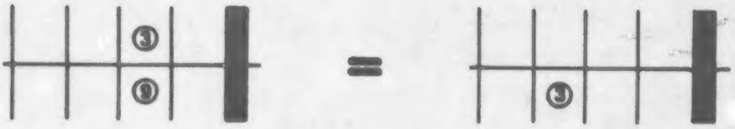
> One checker on an upper square and three checkers on the square directly below may be traded for one checker on the next lower square to the left.



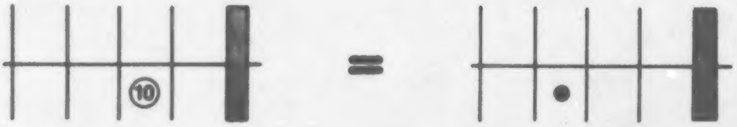
# N 31

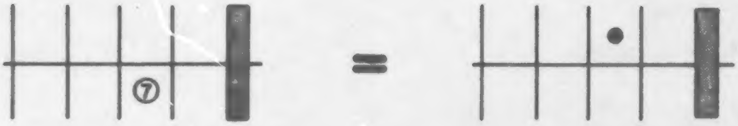
Direct the class to observe other trades that result from this basic trade. For example,



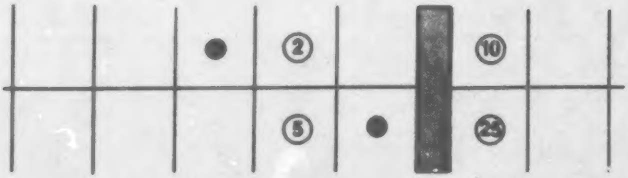


Also review these trades.





Display this configuration of checkers.



T: Write this number on your paper.

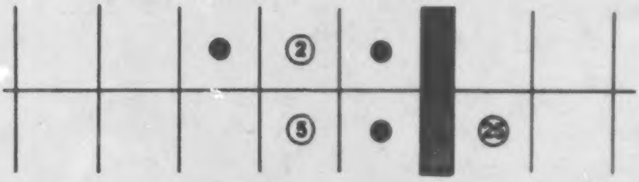
Let the students work independently for a few minutes.

S: 900.5.

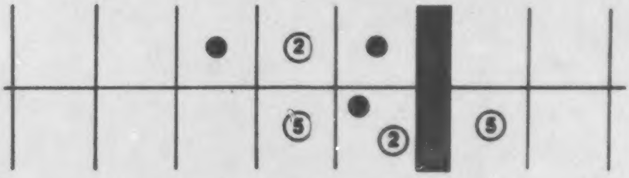
T: Let's check by making some plays to show the same number with fewer checkers.

Call on students to make plays.

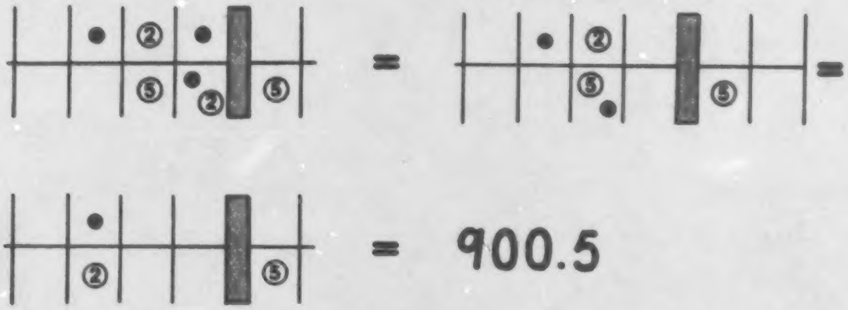
S: Ten checkers on the 0.7-square is the same as one checker on the 7-square.



S: Twenty checkers on the 0.1-square is the same as two checkers on the 1-square.

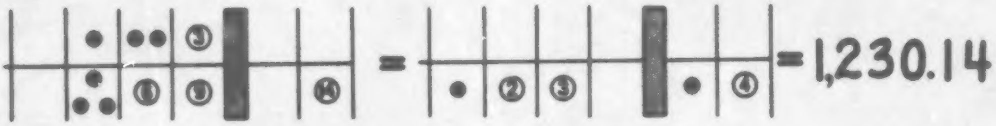


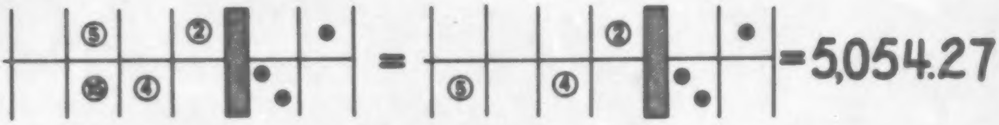
Invite students to continue to make plays. One possible sequence of plays is illustrated below.



# N 31

Continue this decoding activity using the two configurations below or others more appropriate for the numerical level of your class.



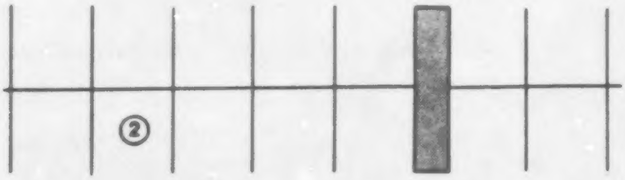


Exercise 2

T: Seven people wish to share $2,000.00 as equally as possible among themselves. How much do you think each person will receive?

Let the students make estimates and record them on the board so that you may refer to them at the conclusion of this activity.

Clear the 7-abacus and display 2,000.

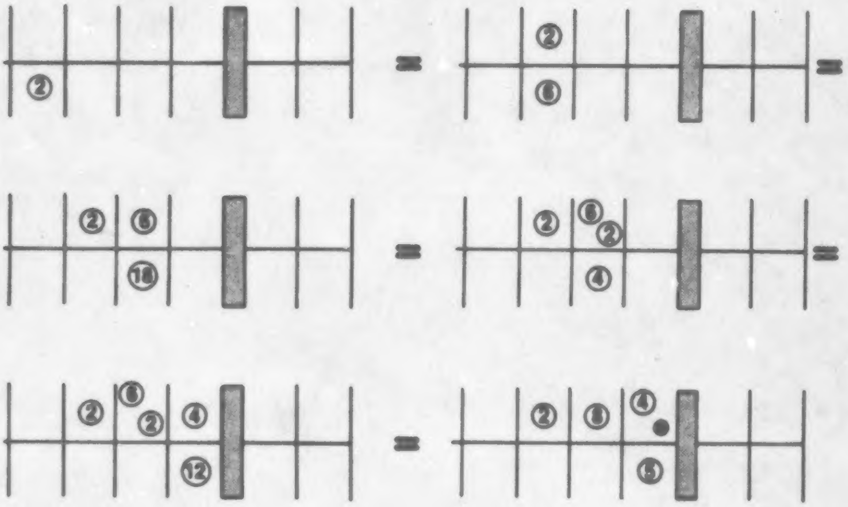


T: How could we use the 7-abacus to solve this problem?

Let the students have an opportunity to suggest showing 2,000 with checkers on the upper squares as much as possible. If this suggestion is not forthcoming, make it yourself.

T: Who can make a play?

Invite students to make backward plays. A possible sequence of plays is shown below.



At this point some students may insist that the problem is solved.

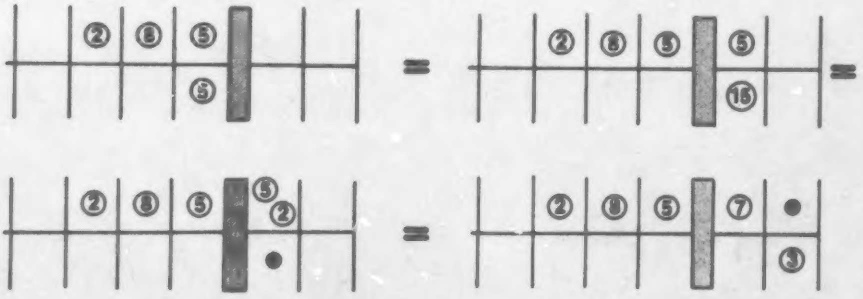
T: How much money should each person receive?

S: \$285 with \$5 left-over, because (7 x 285) + 5 = 2,000.

T: That's correct to the nearest dollar, but what about the \$5 left over.

S: Seven people can share the \$5 with each person receiving between 70¢ and 80¢.

T: Good. Let's continue to make plays on the 7-abacus.



# N 31

T: The abacus shows 2,000 in a special way. On you paper, write another name for 2,000 as suggested by the abacus.

Allow a minute or two for the students to work independently. Then write a number story on the board as a student announces:

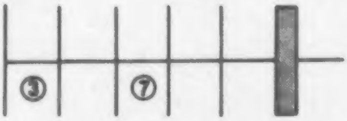
S: 2,000 = (7 x 285.71) + 0.03.

S: So each person should receive $285.71 with 3 cents left over.

Compare the students' estimates to $285.71.

Repeat this activity using other numbers; for example,

30,700 =



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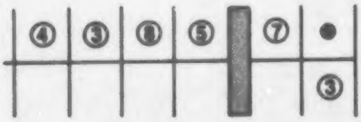
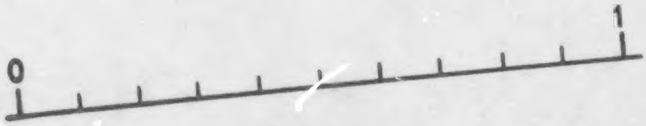


= (7×4,385.71)+0.03

Exercise 3

NOTE: This is an optional exercise for better students.

Draw this part of the number line on the board.



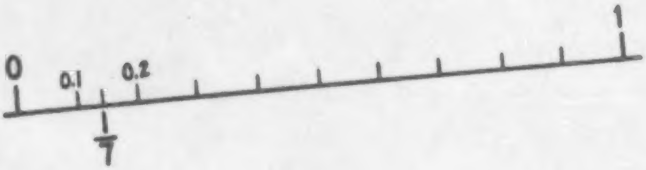
**T:** Who can show approximately where $\frac{1}{7}$ is on the number line?

**S:** $7 \times 0.1 = 0.7$ and $7 \times 0.2 = 1.4$, so $\frac{1}{7}$ is between 0.1 and 0.2.

**T:** Is $\frac{1}{7}$ closer to 0.1 or to 0.2?

**S:** $\frac{1}{7}$ is closer to 0.1, because $7 \times 0.1$ is closer to 1 than is $7 \times 0.2$.

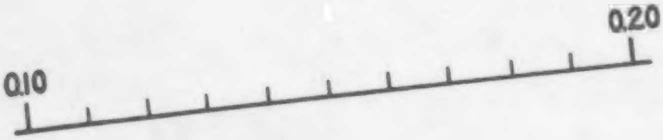
Ask a student to put a mark for $\frac{1}{7}$ on the number line.



**T:** How could we better locate $\frac{1}{7}$?

**S:** Divide the segment between 0.1 and 0.2 into ten segments of the same length.

Draw this "magnified" version of that portion of the number line on the board.



**T:** Where is $\frac{1}{7}$?

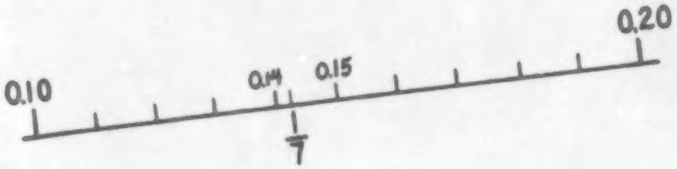
**S:** $\frac{1}{7}$ is between 0.14 and 0.15, because $7 \times 0.14 = 0.98$ and $7 \times 0.15 = 1.05$.

# N 31

**T:** Yes. Is $\frac{1}{7}$ closer to 0.14 or to 0.15?

**S:** $\frac{1}{7}$ is closer to 0.14, because $7 \times 0.14$ is closer to 1 than is $7 \times 0.15$.

Invite a student to label the marks for 0.14, 0.15, and to put a mark for $\frac{1}{7}$ on the number line.



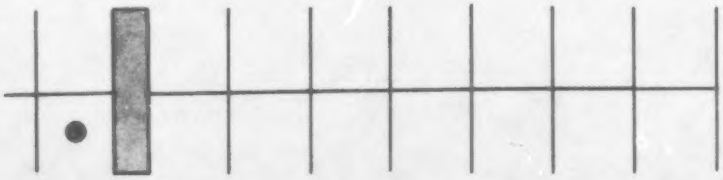
**T:** We could continue in this manner to get closer and closer to the location of $\frac{1}{7}$. Can you think of another way to find closer and closer decimal approximations for $\frac{1}{7}$?

Let the students make suggestions.

**S:** $\frac{1}{7}$ is $1 \div 7$, so we could divide.

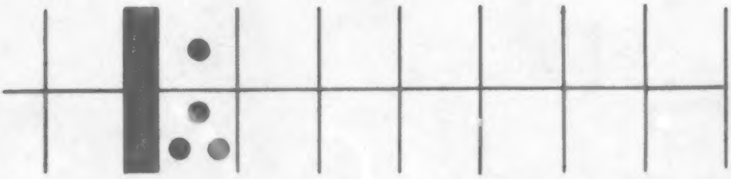
**T:** Very good. How can we perform this calculation using the 7-abacus?

Display 1 on the 7-abacus.

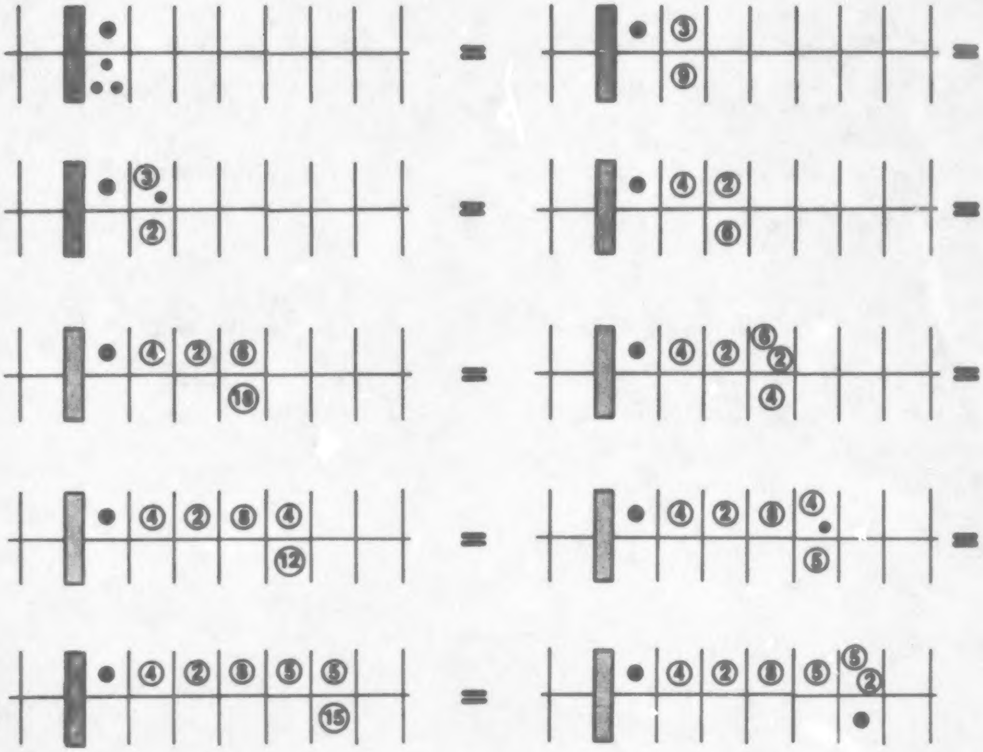


**T:** We want to show 1 on the 7-abacus with checkers on the upper squares as much as possible. Who can make a play?

**S:** One checker (on the 1-square) trades for one checker (on the 0.7-square) and three checkers (on the 0.1-square).



Continue making backward plays until you reach the final configuration shown below.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# N 31

At this point some students may observe that since there is one checker on a lower square the process will repeat itself over and over. If for some reason no student makes this observation, do so yourself.

T: What will happen if we continue to make backward plays on the 7-abacus?

S: "142857" will repeat on the upper squares over and over.

T: Good. Which number is on the abacus?

S: 1.

T: Look carefully at the abacus. On your paper, write another name for $\frac{1}{7}$ as suggested by the abacus.

Give the students a minute or two to work independently. Invite a student to write some names for $\frac{1}{7}$ on the board.

$$\frac{1}{7} = 0.142857142857\ldots = 0.\overset{*}{\overline{142857}}$$

T: Very good. $\frac{1}{7}$ is like $\frac{1}{3}$. We cannot write decimal names for them, but we can write this kind of nickname.

Hold up a hand-calculator.

T: How can we display $\frac{1}{7}$ on the calculator?

S: $\frac{1}{7}$ is 1 ÷ 7, so press $\boxed{1}$ $\boxed{\div}$ $\boxed{7}$ $\boxed{=}$.

Press the appropriate keys, then show the display to the class.

T: The display shows "0.1428571". What should happen if I multiple this number by 7?

S: $\frac{1}{7} \times 7 = 1$, so you should get 1 on the display.

S: You won't get 1 because 0.1428571 is not exactly $\frac{1}{7}$.

Press $\boxed{x}$ $\boxed{7}$ $\boxed{=}$ and show the result to the class.

T: 0.9999997[*] is on the display.

S: That's close to 1 but not equal to 1.

T: Yes, the calculator can only give an approximation to 1 ÷ 7 because it can only show seven places but with the 7-abacus we can indicate how the digits go on and on.

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[*] Check the calculator you are using; some calculators behave differently in this situation.

# 7

# CARTESIAN GRAPHS

The lessons of this chapter introduce the idea of graphing relations in a two dimensional coordinate system. The resulting graphs are called "Cartesian graphs" because one commonly refers to the two dimensional coordinate system as the "Cartesian plane".* Graphs for several kinds of relations provide opportunities to make observations which distinguish the Cartesian graphs of functions from those of non-functions.

CAPSULE LESSON SUMMARIES

| | |
|---|---|
| Section 7.1: | Introduce the notion of a Cartesian graph of a relation and graph several relations including the + 2 function, the "distance-from-0" function, and the 2 x function. (Worksheets N63, N64) |
| Section 7.2: | Review the notion of a Cartesian graph and graph several relations including the "is less than" relation. Introduce a red (out) - blue (in) convention to indicate points which do not or do belong to a graph. (Worksheet N65) |

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* "Cartesian" is a name honoring the French mathematician René Descartes (1596 - 1650) for his unification of algebra and geometry in the creation of analytic geometry.

# N 26

## 7.1 Cartesian Graphs #1

CAPSULE LESSON SUMMARY

Introduce the notion of a Cartesian graph of a relation and graph several relations including the +2 function, the "distance-from-0" function, and the 2x function.

MATERIALS

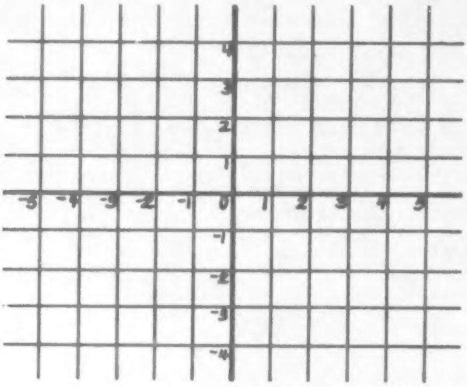
Teacher: Colored chalk; grid board (optional)
Student: Colored pencils; Worksheets N63 and N64

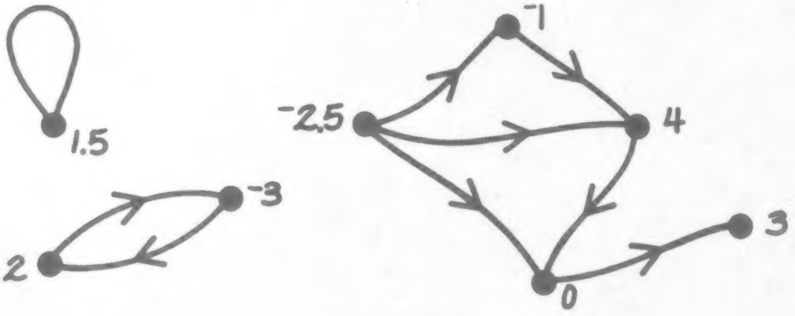
NOTE: If a grid board is available, you may prefer to use it rather than to draw a grid on the chalkboard as described in this lesson.

DESCRIPTION OF LESSON

Exercise 1

Draw this grid and arrow picture on the board.



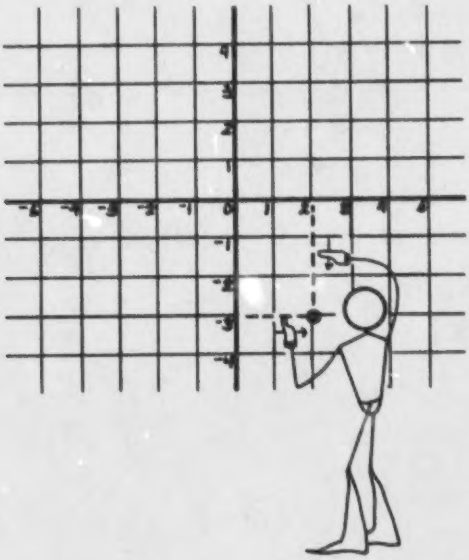


T (tracing the arrow from 2 to ⁻3): This arrow is for which couple of numbers?

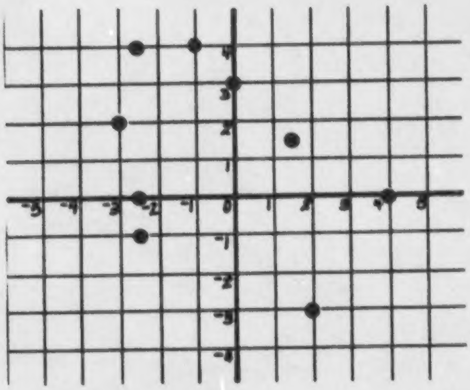
S: (2, ⁻3), because 2 is the starting number and ⁻3 is the ending number.

T: Good. Where is the point on the grid for the couple (2, ⁻3)?

Invite a student to explain how to locate the point.

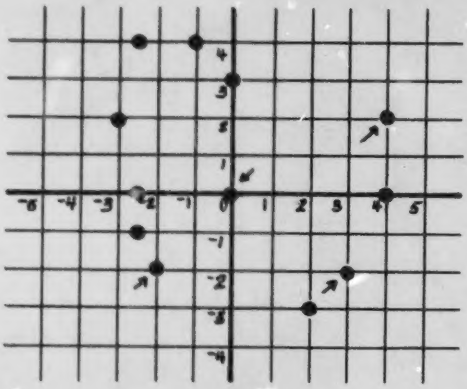


Continue this activity by asking students to locate points for the eight remaining arrows. Emphasize that the starting number is located along the horizontal axis and the ending number is located along the vertical axis. Upon completion of this activity, there should be nine dots on your grid as shown below.



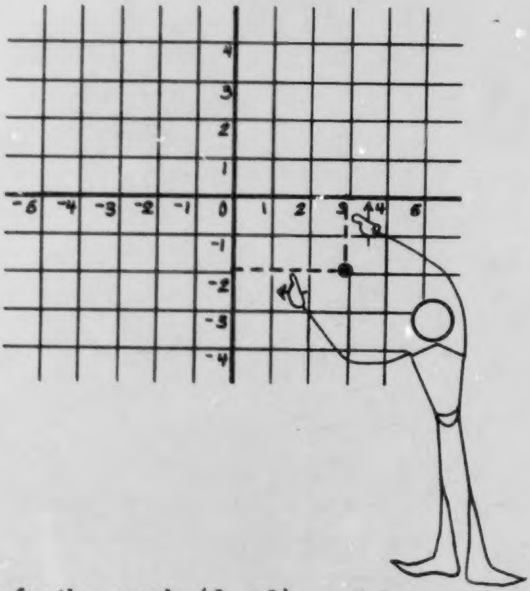
# N 26

Reverse the situation by adding one at a time four blue dots (as indicated below) to your grid picture and asking students to draw the corresponding arrows.



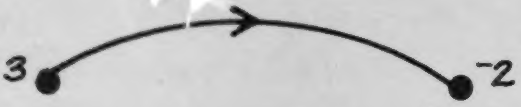
$^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5
4 3 2 1 $^{-}1$ $^{-}2$ $^{-}3$ $^{-}4$

Request that students explain how to draw the appropriate arrows.

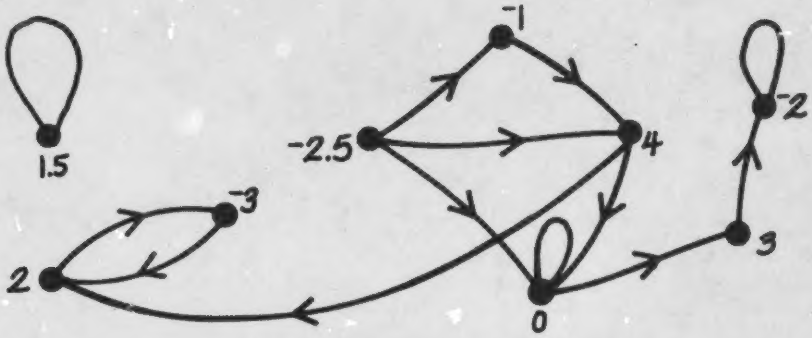


$^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5
4 3 2 1 $^{-}1$ $^{-}2$ $^{-}3$ $^{-}4$

S: This point is for the couple (3, $^{-}2$), so I draw an arrow starting from 3 and ending at $^{-}2$.

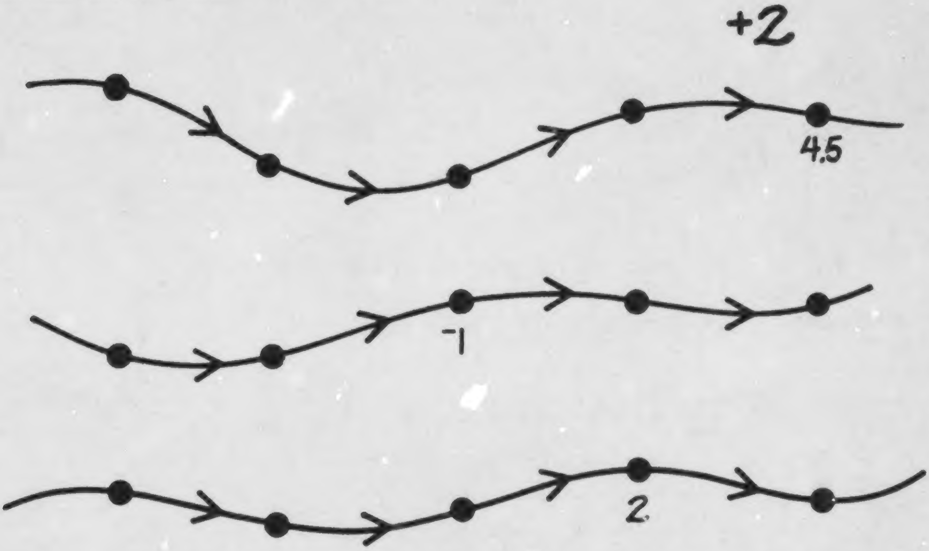


The completed arrow picture is shown below.



Exercise 2

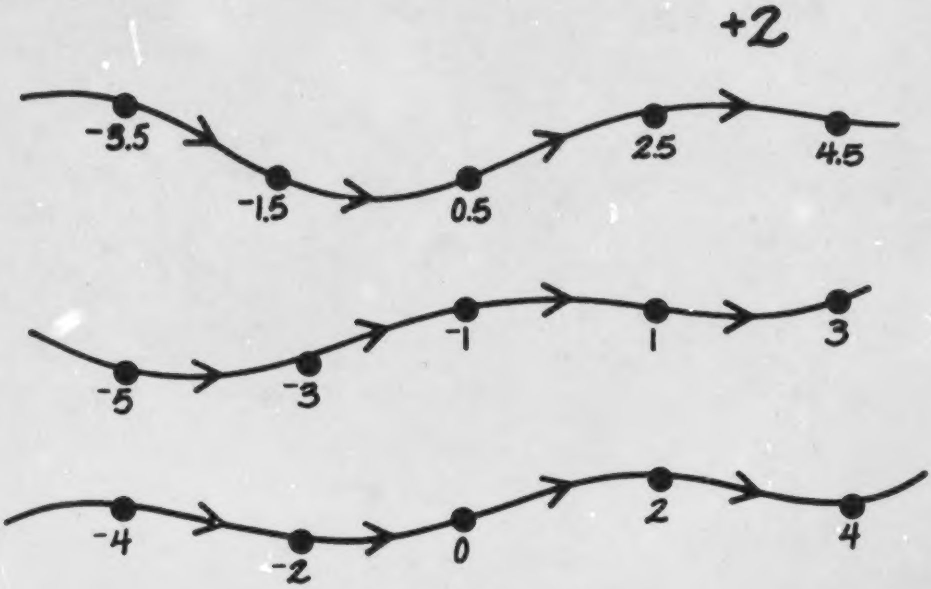
Draw this arrow picture on the board.



T: This is part of an arrow picture for the + 2 function.

Ask students to label the dots.

# N 26



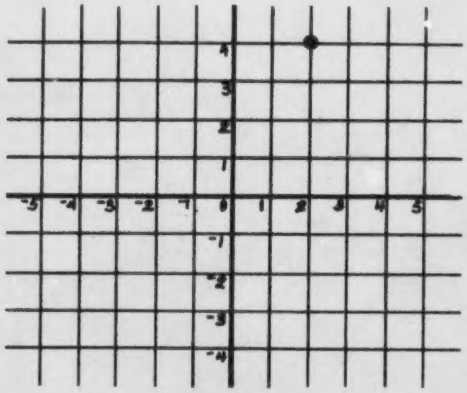
T: Do you notice anything interesting about this picture?

S: Even numbers are all on one road and odd numbers are on another road.

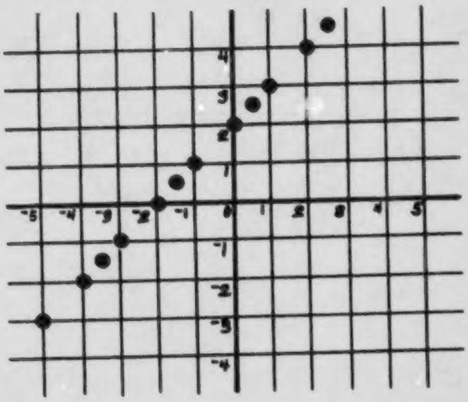
Let the class comment freely about this picture.

T: Let's locate some points which correspond to the arrows of the +2 function. We'll see if there are any interesting patterns in its grid-picture. This arrow (point to the arrow from 2 to 4) is for the couple (2, 4). Where is the point on the grid for this arrow?

Ask a student to draw a red dot at the point for (2, 4).



Invite students to locate points for two or three more arrows. Then distribute Worksheet N6, and ask them to find on their grid a point for each arrow of the +2 function on the board. When it is obvious that most students have finished, complete the picture at the board.



**T:** Did we draw a dot for every arrow of the +2 function?

**S:** No, there are many other arrows that could be drawn; for example, an arrow starting at 100 and ending at 102.

**T:** Yes. Could we draw dots at all the possible points for arrows?

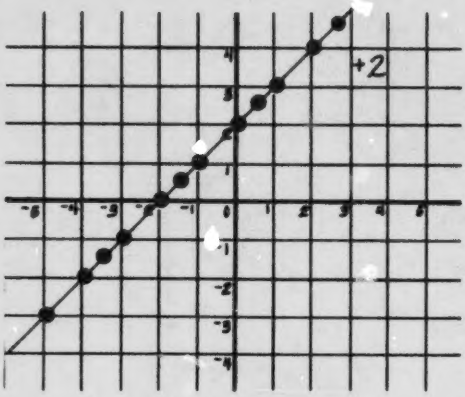
**S:** No, there is an endless number of points.

**T:** Where are the points for the arrows that are missing?

**S:** Along this line; they are the points on a line connecting the ones we have marked already.

# N 26

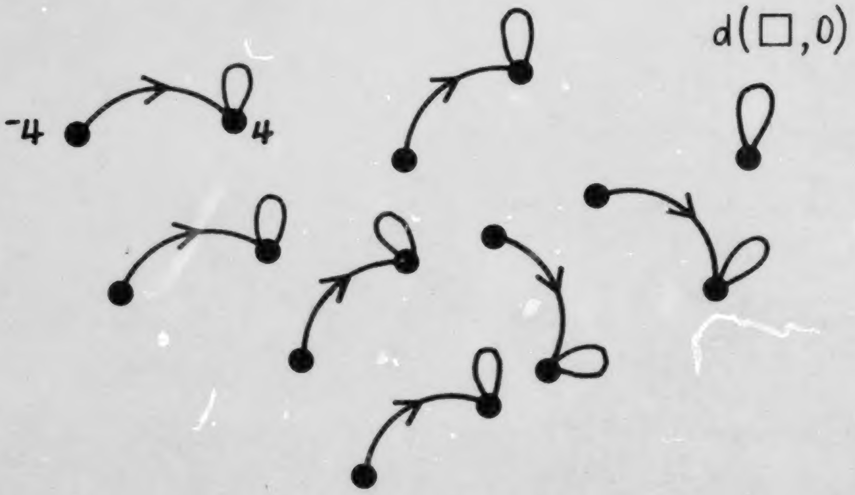
Draw the line in red and label it "+ 2".



T: This line is the Cartesian graph of the + 2 function.

Exercise 3

Draw this arrow picture on the board.

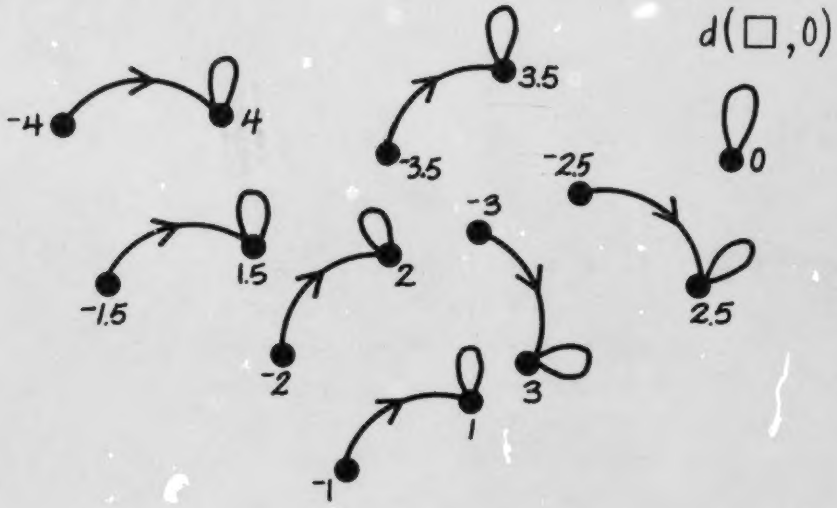


T: This is part of an arrow picture for the "distance-from-0" relation. ‾4 points to 4 (trace the arrow from ‾4 to 4) because ‾4 is at a distance of 4 away from 0 on the number line.

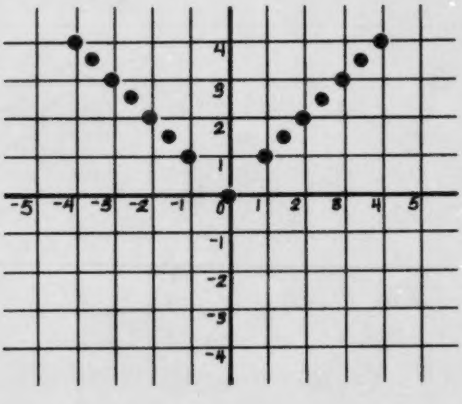
If necessary, demonstrate this fact on a number line.

T: Also, 4 is at a distance of 4 from 0 on the number line, so 4 points to 4 (trace the loop at 4).

Invite students to provide labels for the remaining dots. Occasionally, ask for an explanation of an arrow or a loop. Encourage the students to use numbers that can be located in the grid that is on the board. Your arrow picture might look like this upon completion.



Distribute Worksheet N64 and ask the students to draw the Cartesian graph for the "distance-from-0" relation. When most students have finished, draw a graph at the board.



# N 26

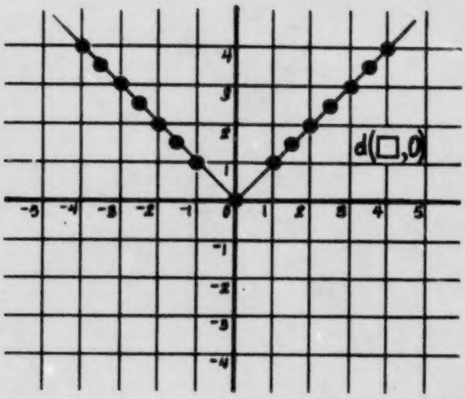
**T:** Is this a complete picture of the "distance-from-0" relation?

**S:** No, there is an endless number of arrows that could be drawn and so an endless number of points to be found.

**T:** What do you think the picture would look like if we could draw dots at all possible points?

**S:** There would be two slanting lines.

Draw the graph in blue and label it.



**T:** Look at the arrow picture and its graph. Is the "distance-from-0" relation a function?

Allow a few minutes for the class to consider the question and perhaps to review the notion of function.

**S:** Yes, there are no guilty points in the arrow picture. Exactly one arrow starts from each dot.

**T:** Yes; but can we tell by looking at its Cartesian graph if the relation is a function?

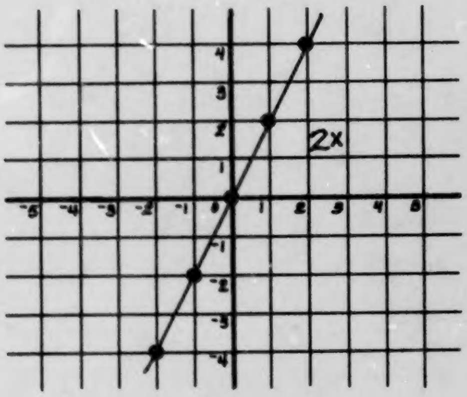
S: Yes; there is exactly one dot on any vertical line.

Accept any reasonable explanation even if it is not exactly like the above.

T: Do you notice anvthing else that is interesting about this function?

S: If you fold along the vertical line through 0, the blue lines fold on top of each other.

If time permits, ask the students to draw the Cartesian graph for the 2 x function.



# N 34

## 7.2 Cartesian Graphs #2

CAPSULE LESSON SUMMARY

Review the notion of a Cartesian graph and graph several relations including the "is less than" relation. Introduce a red (out)- blue (in) convention to indicate points which do not or do belong to a graph.

MATERIALS

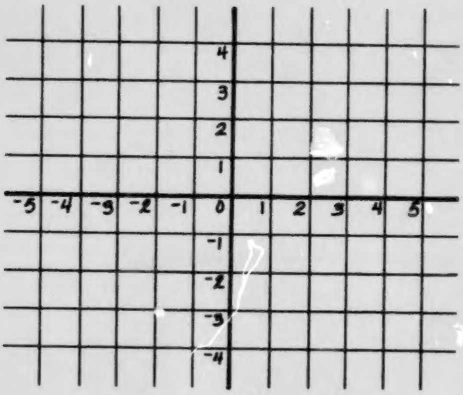
Teacher: Colored chalk; grid board (optional)
Student: Paper; colored pencils; Worksheet N65

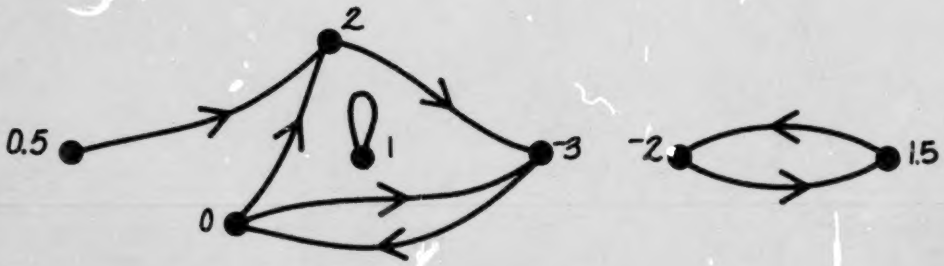
NOTE: If a grid board is available, you may prefer to use it rather than to draw a grid on the chalkboard as described in the lesson.

### DESCRIPTION OF LESSON

#### Exercise 1

Draw this grid and arrow picture on the board.



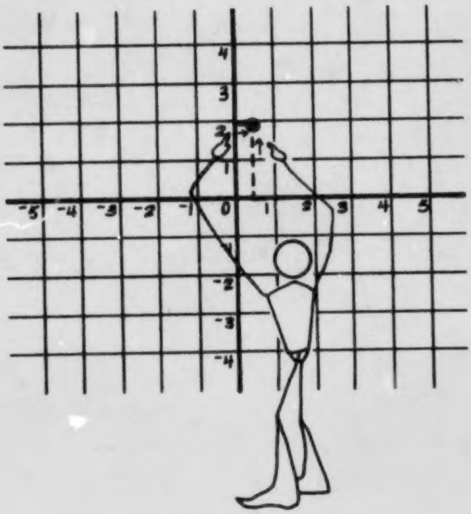


T (tracing the arrow from 0.5 to 2): This arrow is for which couple of numbers?

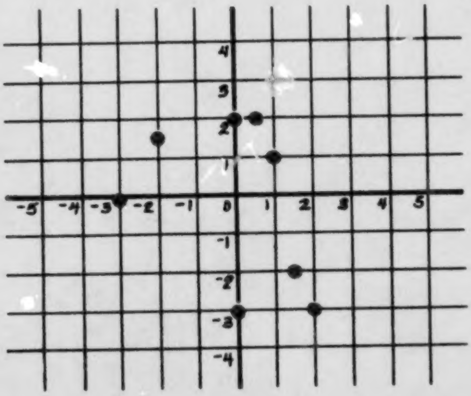
S: (0.5, 2), because 0.5 is the starting number and 2 is the ending number.

T: Yes. Where is the point on the grid for the couple (0.5, 2)?

Invite a student to explain how to locate the point.



Continue this activity by asking students to locate points for the seven remaining arrows. Emphasize that the starting number is located along the horizontal axis and the ending number is located along the vertical axis. Upon completion of this activity, there should be eight dots on your grid as shown below.



T (pointing to the arrow picture): Is this relation a function?

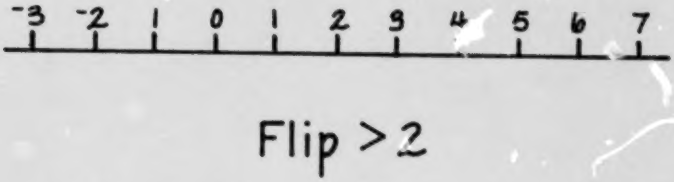
S: No, 0 is a guilty point; there are two arrows starting at 0.

T (pointing to the grid): How can we tell by looking at its Cartesian graph that the relation is not a function?

S: Look at the vertical line through 0; there are two points on it corresponding to the two arrows that start at 0.

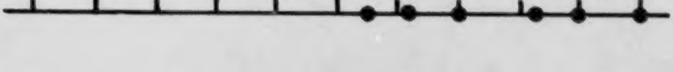
Exercise 2

Draw this part of the number line on the board. Write a number story about Flip below the line.



T: Let's draw blue dots on the number line for numbers that Flip could be.

Invite several students to draw blue dots. After several dots are drawn, students should notice that all of the dots are to the right of 2. Your number line might look like this.



T: Which is the smallest number Flip could be?

Of course, there is no smallest number that Flip could be but it may be the first time some students have given much thought to the question. Lead a class discussion to observe that there is no smallest number. For example,

S: Flip could be 2.1.

T: Can you think of a smaller number that Flip could be?

S: 2.01.

T: Is this the smallest number Flip could be?

S: No.

T: Can you find a smaller number?

S: 2.00 . . . 1 with 125 zeroes.

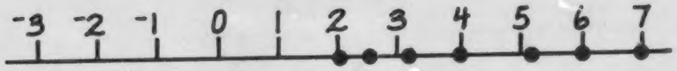
T: Is this the smallest?

S: No, not even if there were 1,000 zeroes before 1.

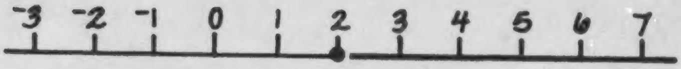
S: There is no smallest number that Flip could be.

S: The numbers get closer and closer to 2, but Flip cannot be 2.

T: Very good. To show that Flip cannot be 2, I will draw a red dot at 2.



**T:** Since Flip could be any number to the right of 2, I will shade that part of the number line blue.



**T:** Let's agree that blue shows numbers that Flip could be and, when we need to make the picture clear, red shows numbers that Flip could not be.

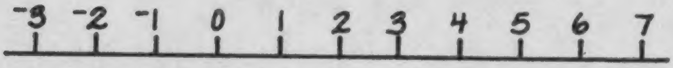
Erase the number story and the red dot at 2 on the number line.

**T:** Look again at the number line. Suppose I draw a blue dot at 2. Which number story would this be for?

Allow a minute or two for the students to consider the question.

**S:** Flip could be any number larger than or equal to 2.

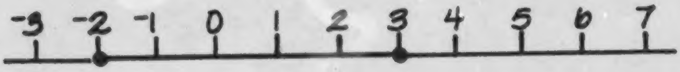
Erase and redraw the number line. Write a new number story about Flip below the line.



$^-2 < \text{Flip} < 3$

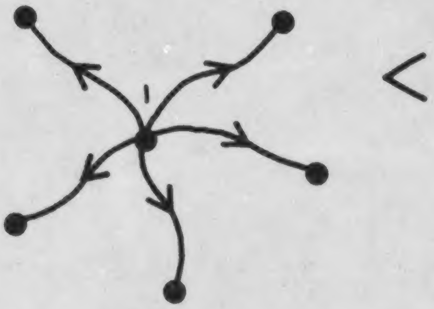
**T:** Using our red-blue convention, indicate which numbers Flip could be.

Give the students a few minutes to work independently. Then invite a student to draw the picture on the board.



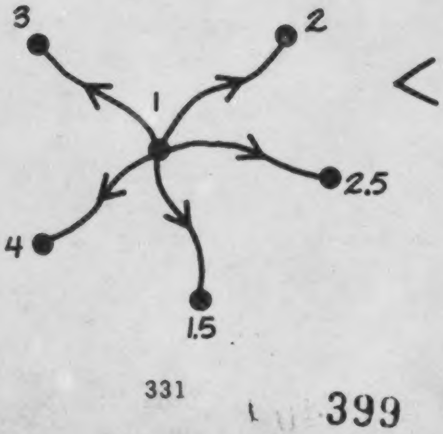
Exercise 3

Distribute Worksheet N65 and draw this arrow picture on the board.



T: The blue arrows are for the relation "is less than". How should we label the dots?

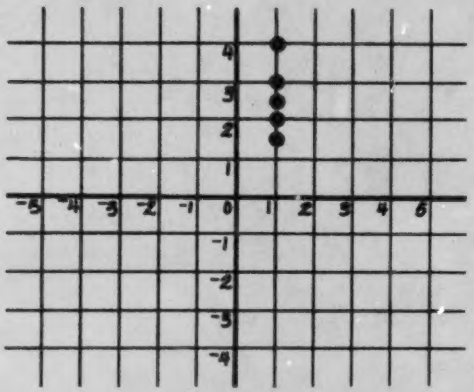
Invite some students to label the dots.



# N 34

**T:** On your grid, find the points for these arrows.

Give the students time to work independently on this problem. Ask some students to draw the dots on the demonstration grid.



**T:** Could we draw any more arrows starting at 1?

**S:** Yes, an endless number of them.

**T:** What can we say about the ending numbers of these arrows?

**S:** Each ending number will be larger than 1.

**T:** Yes, but is there a smallest ending number?

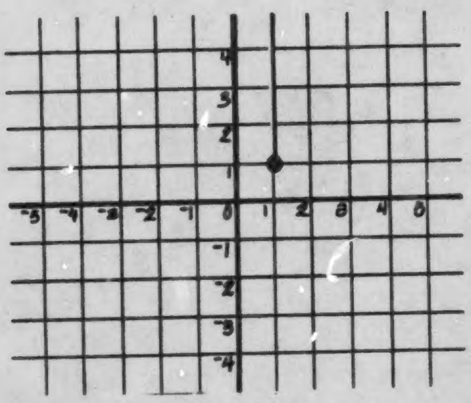
**S:** No, it's like the first problem about Flip. There is no smallest number. The numbers are larger than 1 but can get closer and closer to 1.

**T:** How can we show this situation on the grid?

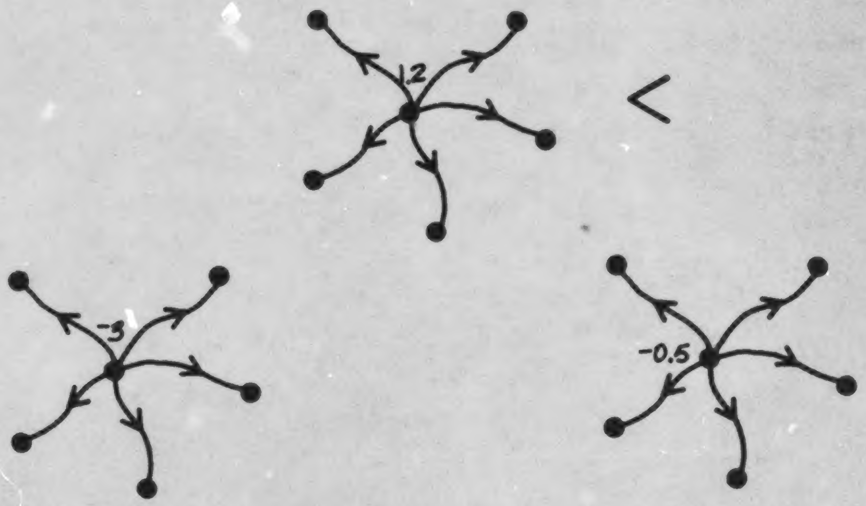
**S:** It's like the problem of showing the numbers Flip could be on the number line when we had "2 < Flip". Imagine a vertical number line. Color in blue to show numbers larger than 1.

S: Draw a red dot at 1 (actually, (1, 1)) to show that 1 is not less than 1.

Invite a student to draw the picture.

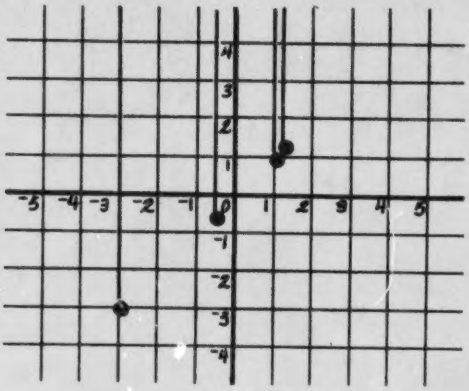

4
3
2
1
-5
-4
-3
-2
-1
0
1
2
3
4
5
-1
-2
-3
-4

Add this information to your arrow picture.



# N 34

In a similar manner, let the students analyze the situation for all of the arrows starting at ⁻3; at 1.2; and at ⁻0.5. When most students have finished, ask some students to put their drawings on the board.

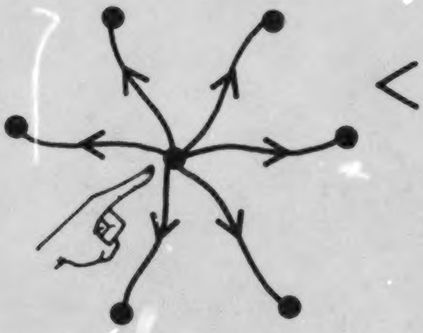


T: Look at the grid picture. Do you notice anything interesting?

Let the class make some observations.

Perhaps students will observe that the red dots lie on a line which goes through (0, 0) and includes all the couples of the form "(n, n)". They may also comment that the entire part of a line (open half-line) above each red dot is colored blue. Accept any reasonable comments.

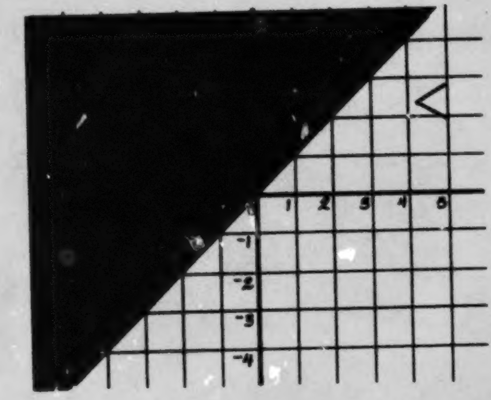
Draw a new piece in the arrow picture and point to the center dot.



T: Any number could be here. On your grid, find other couples that belong in the "is less than" relation by thinking about lots of possibilities for numbers at this dot. Try to find all the couples that belong in the "is less than" relation.

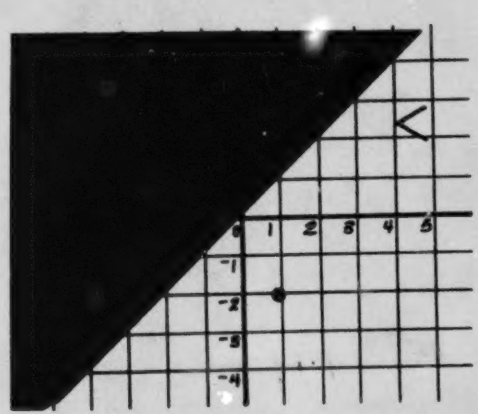
Provide time for students to work independently on this problem. Walk around and observe the students' pictures. When several students have the full solution, invite one student to draw the picture on the board.

S: The blue lines run together like this.



T (tracing the blue region): This region is the Cartesian graph of the "is less than" relation. Where is a point which does not belong to the "is less than" relation?

Ask a student to draw an appropriate red dot.



# N 34

| | |
|---|---|
| T: | What is the couple of numbers for this point? |
| S: | (1, ⁻2). |
| T: | What do you notice about these numbers? |
| S: | 1 is larger than ⁻2. |

Repeat this activity several more times. The students should conclude that for couples of numbers which do not belong to the "is less than" relation the first number is equal to or greater than the second number.

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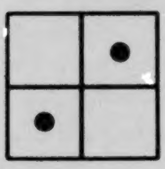
# APPENDIX: STANDARD CONFIGURATION AND STANDARD MINICOMPUTER MOVES

A number can be represented by putting checkers[*] on the squares of the Minicomputer. Each square has a numerical (positional) value, so that the number represented by a checker depends on which square the checker is placed. If several checkers are put on the Minicomputer, the number shown is the sum of the numbers represented by the checkers. A number can be put on the Minicomputer in a variety of ways, but the representation which uses at most one checker on each square and uses checkers to represent a digit 9 or less on each board is usually the easiest to read. In this case, we say that the number is in standard configuration. Standard configurations for the numbers 1 - 9 soon become as familiar to the students as the usual numerals so that they no longer need to do any mental calculation for such configurations.

For example,



. . . is seen as a name for 6 without doing the calculation "4 + 2 = 6".

Doing calculations on the Minicomputer involves putting one or more checkers on the various squares and making "plays" to get a specific configuration. In this context, a play is an exchange of checkers in a configuration which does not change the number represented. Some plays are natural to the design of the Minicomputer; we call these standard or usual plays (these are described in detail in this section). There are advantages to establishing such a set of plays:

---

[*] In this appendix we refer only to whole numbers and, therefore, use only regular (positive) checkers.

- They become routine, and a large group of students with varying abilities can follow a calculation without needing to verify every play which is made.
- They lead to standard configurations of numbers.

These are the standard Minicomputer moves.

1 + 1 = 2
10 + 10 = 20
100 + 100 = 200
1,000 + 1,000 = 2,000

two checkers on the white square
of a board
=
one checker on the red square
of the same board

2 = 1 + 1
20 = 10 + 10
200 = 100 + 100
2,000 = 1,000 + 1,000

one checker on the red square
of a board
=
two checkers on the white square
of the same board

2 + 2 = 4
20 + 20 = 40
200 + 200 = 400
2,000 + 2,000 = 4,000

two checkers on the red square
of a board
=
one checker on the purple square
of the same board

4 = 2 + 2
40 = 20 + 20
400 = 200 + 200
4,000 = 2,000 + 2,000

one checker on the purple square
of a board
=
two checkers on the red square
of the same board

$4 + 4 = 8$
$40 + 40 = 80$
$400 + 400 = 800$
$4,000 + 4,000 = 8,000$

two checkers on the purple square
of a board
=
one checker on the brown square
of the same board

$8 = 4 + 4$
$80 = 40 + 40$
$800 = 400 + 400$
$8,000 = 4,000 + 4,000$

one checker on the brown square
of a board
=
two checkers on the purple square
of the same board

$8 + 2 = 10$
$80 + 20 = 100$
$800 + 200 = 1,000$
$8,000 + 2,000 = 10,000$

one checker on each of the brown
and red squares of a board
=
one checker on the white square
of the next board to the left

$10 = 8 + 2$
$100 = 80 + 20$
$1,000 = 800 + 200$
$10,000 = 8,000 + 2,000$

one checker on the white square
of a board
=
one checker on each of the brown
and red squares of the next board
to the right*

---

* This provides the link that motivates the introduction of the decimal numbers. In particular, we have that one checker on the white square of the ones' board equals one checker on each of the brown and red squares of a special board (the tenths' board - or as it is often referred to in the lessons, the dimes' board) located to the right of the ones' board and separated from it by a special barrier.

# WORKBOOKS

Comprehensive School Mathematics Program

# Mathematics for the Intermediate Grades
# Part V

Workbooks

CEMREL, Inc.

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# WORKBOOK

## INTRODUCTION

There are many opportunities for the students to work individually during the course of the lessons described in the four content strands. In the workbook strand, however, it is this individualized work which becomes the chief end of most of the lessons. It has been our goal in this strand to provide the students with opportunities

- to review many of the ideas they have met in the content strand;
- to apply their acquired knowledge to new situations requiring various kinds of strategic thinking;
- to learn how to read and use mathematics workbooks and story-workbooks.

The W-strand includes the following workbooks.

Arcade of Problems #1
Arcade of Problems #2
Arcade of Problems #3
Arcade of Problems #4
Arcade of Problems #5
Arcade of Problems #6

Each of these six workbooks has problems of varying levels of difficulty all in one booklet. Approximately the first ten pages of each booklet are easy problems, the next ten to twelve pages are average problems, and the last ten pages are challenging problems.

The W-strand also includes the following storybook.

ELECTION IN THE NUMBER WORLD

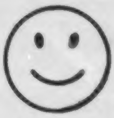
We suggest that you devote one day a week to this strand as indicated on the suggested schedule of lessons given in the General Introduction. We also recommend that you follow this schedule more or less as given, even if many of your students have not finished a particular workbook after the two periods that are allotted to it. As a general rule, students at this age do not enjoy remaining on the same topic for too long. It is a basic principle of the CSMP spiral approach that a variety of experiences is preferable to any attempt to master a given topic by continued "plugging away" at it. This principle applies as much (if not more so) to the Workbook strand as it does elsewhere in the curriculum. If you do feel, however, that more time is needed for a particular workbook, you could allow students to continue their work on that workbook during one (or more) of the adjustment days or during their free time.

The placement of the various workbooks in the suggested schedule of lessons was made keeping in mind where lessons from the other content strands occur. The types of problems in each workbook reflect the topics covered in earlier lessons and assume information has previously been introduced. Hence, you are warned to be particularly careful when considering possible modification of this suggested schedule. Whatever changes you might like to make, we strongly urge that you use one day each week for workbooks.

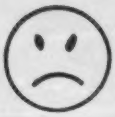
For the storybook ELECTION IN THE NUMBER WORLD, there are three collective lessons that accompany it. There are also short collective (warm-up) lessons suggested for some of the "Arcade of Problems" workbooks. Following any such suggestions for a particular workbook, there is an answer key for that workbook. All answer keys are presented in reduced format, each page usually containing reduced copies of four of the pages of the original booklets, with solutions written in.

For each workbook we suggest that all students start work at the easiest level of difficulty and then work through as many pages as they can handle during the two scheduled lessons. You can expect that, under normal circumstances, about two-thirds of the students will work through about 20 to 25 pages, and about one-third will reach the most challenging problems at the end of the booklet.

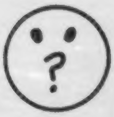
The workbooks provide an excellent instrument with which you can assess the progress of your students on a week-by-week basis. To use this instrument effectively certainly does not require that you check every page and every problem for each student. You should develop the sort of procedure for checking students' work with which you are most comfortable. For example, you might check one or more specific pages, discuss some particular mistakes with individual students, and let them correct their work before continuing. Or you might simply look carefully at the first few pages of the workbooks to see that the students understand the general idea of the problems. In our experience, an effective procedure is to indicate on the front cover of each checked workbook an assessment of the students' work on each attempted page. For this purpose, you might use the following symbols.



meaning "well done" or "correct"

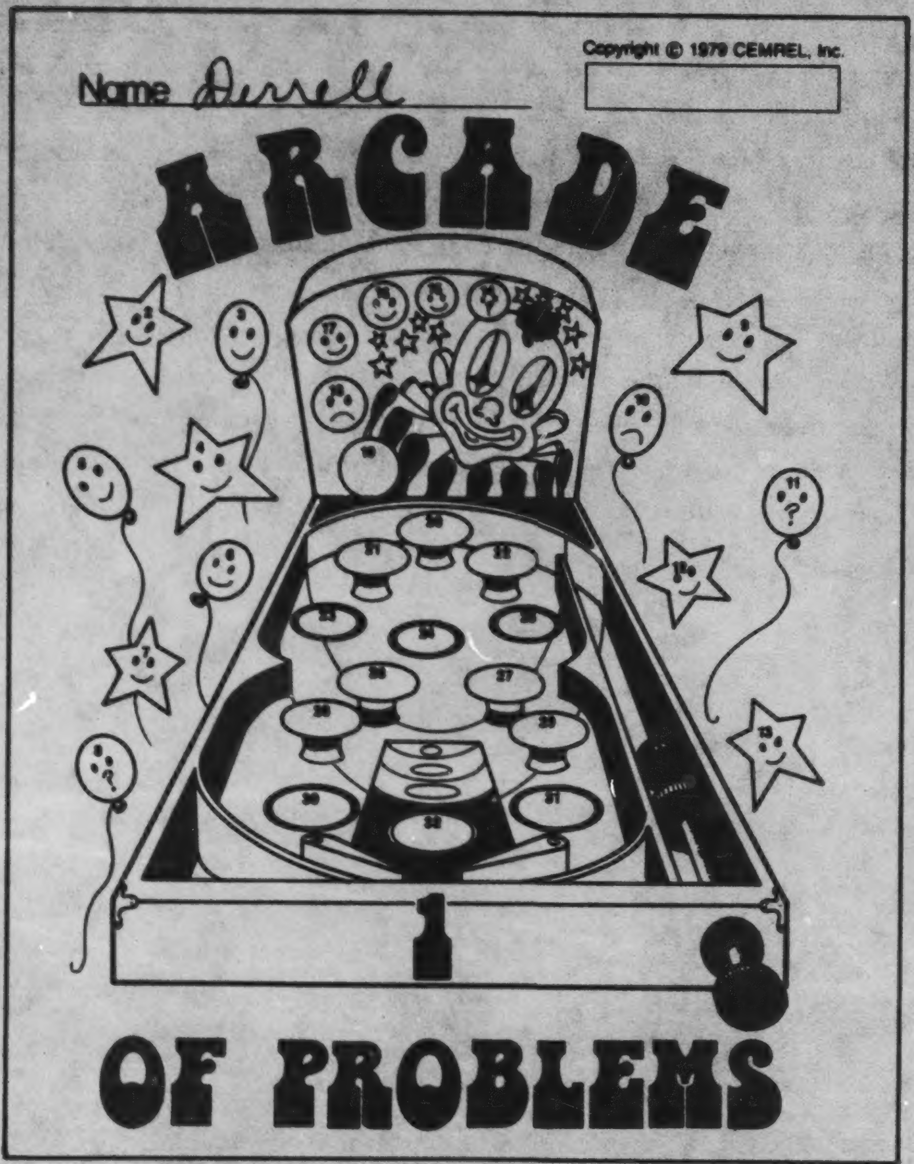


meaning "something wrong, try again"



meaning "did you forget this problem?"
or "do you need a hint?"

On the next page we show the front cover of one students' Arcade of Problems #1 workbook after it has been checked.



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Name Derrell

# ARCADE 18 OF PROBLEMS

**NOTE:** This is the way Derrell's workbook was returned to him at the beginning of the second lesson devoted to this workbook. The blanks in spaces 19-32 indicate that he had not yet worked on these pages. Before Derrell begins on these pages during the second lesson, he should correct his mistakes on pages 10 and 18 and finish his work on pages 8, 11, and 14.

Immediately following this introduction to the W-strand, we provide a progress chart to help you keep track of

- where each of your students is on any particular workbook day;
- how many pages each student has completed under each title;
- your personal degree of satisfaction with the student's work on each workbook.

In particular you should be on the watch to see if, for any of your students, even the easiest problems in the workbooks are too difficult. Such students probably would benefit a great deal if you were able to devote some time with them on the activities suggested in the book "Math Play Therapy".

Here is a list of some of the important points to bear in mind when working in this strand.

- All students start at the beginning of each workbook and progress as far as they can.
- All students begin a workbook on the same day, even if some students have not finished work on the preceding workbook.
- Not all students are expected to complete a given workbook. Only the faster students will reach the most challenging problems. The slower students may succeed to do only the easiest problems although you should not assume that this will automatically be the case—surprises are not at all uncommon in this area.
- Read the introductory material for each workbook and storybook and give the full group lesson(s) when appropriate.

Workbook Progress Chart

| Name | Arcade of Problems #1 | Arcade of Problems #2 | Arcade of Problems #3 |
|---|---|---|---|
| | | | |
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## Workbook Progress Chart

| Name | Arcade of Problems #4 | Arcade of Problems #5 | Arcade of Problems #6 |
|---|---|---|---|
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# ARCADE OF PROBLEMS

# #1

# ARCADE OF PROBLEMS #1

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.# You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems.

---

# See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

# W 1&2

## Lesson 1

Give each student a copy of "Arcade of Problems #1" and let them work on their own.

If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

## Lesson 2

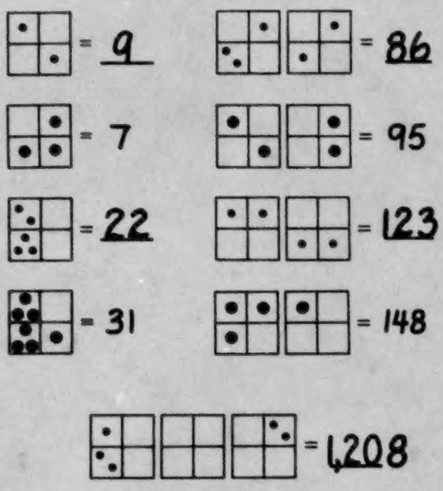
Return the students' copies of "Arcade of Problems #1". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

# Arcade of Problems # 1

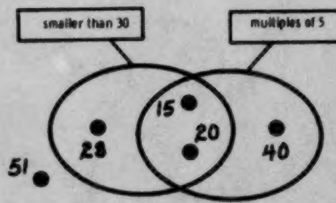
Complete.


= 9
= 86
= 7
= 95
= 22
= 123
= 31
= 148
= 1,208

(Other solutions are possible.)

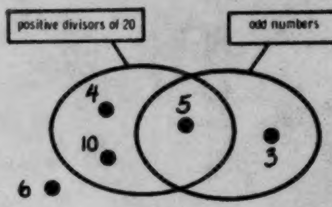
Put each of these numbers in the string picture.

15 20 28 40 51

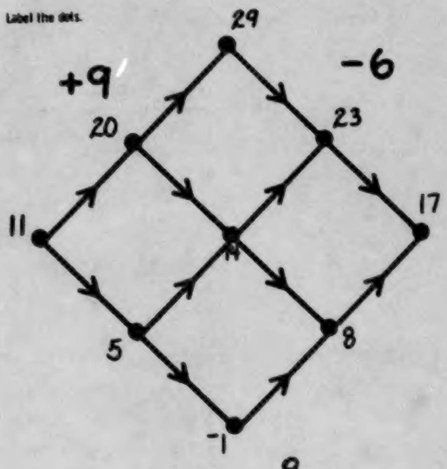

smaller than 30
multiples of 5
15
20
28
40
51

Put each of these numbers in the string picture.

3 4 5 6 10


positive divisors of 20
odd numbers
4
5
10
3
6

Label the dots.


+9
-6
29
20
23
11
14
17
5
8
-1

The smallest even number in this picture is 8.

The largest multiple of 4 in this picture is 20.

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

Add

$$\begin{array}{r} \boxed{5}\,7\,2\,3\,\boxed{8} \\ +\ 9\,\boxed{0}\,\boxed{6}\,7 \\ \hline 6\,\boxed{6}\,3\,0\,5 \end{array}$$

Add

$$\begin{array}{r} 8\,3 \\ 9\,\boxed{6}\,6\,\boxed{7} \\ \boxed{4}\,1 \\ +\ \ 4\,7\,4 \\ \hline \boxed{1}\,\boxed{0}\,2\,6\,5 \end{array}$$

Add

$$\begin{array}{r} 8\,6\,7 \\ 4\,6 \\ +\ 5,4\,3\,7 \\ \hline 6,3\,5\,0 \end{array}$$

Subtract

$$\begin{array}{r} 8\,\boxed{8}\,9 \\ -\ 6\,5\,\boxed{6} \\ \hline \boxed{2}\,3\,3 \end{array}$$

Subtract

$$\begin{array}{r} 7\,5\,2\,4 \\ -\,\boxed{4}\,6\,8\,\boxed{0} \\ \hline 2\,\boxed{8}\,\boxed{4}\,4 \end{array}$$

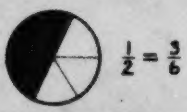
Subtract

$$\begin{array}{r} 1\,4,0\,5\,3 \\ -\ \ 7,2\,4\,8 \\ \hline 6,8\,0\,5 \end{array}$$

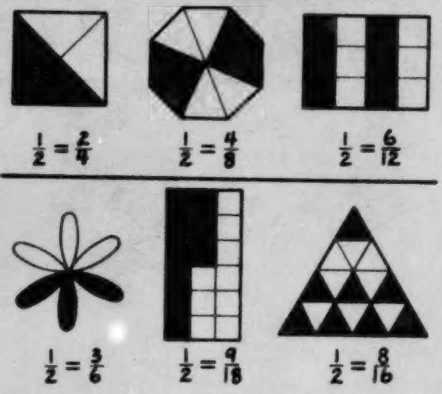
Color one-half of each shape red.

Write another name for $\frac{1}{2}$ as suggested by the picture.

Example



$\frac{1}{2} = \frac{3}{6}$

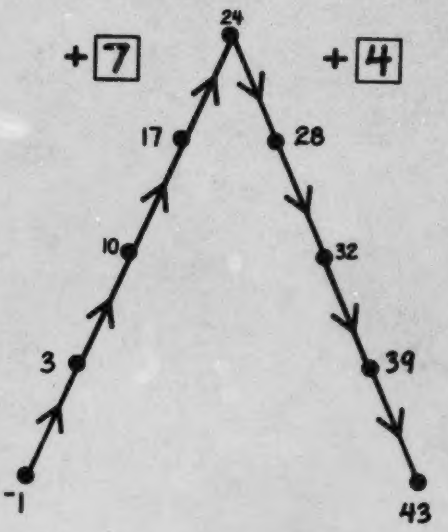


$\frac{1}{2} = \frac{2}{4}$ $\frac{1}{2} = \frac{4}{8}$ $\frac{1}{2} = \frac{6}{12}$

$\frac{1}{2} = \frac{3}{6}$ $\frac{1}{2} = \frac{9}{18}$ $\frac{1}{2} = \frac{8}{16}$

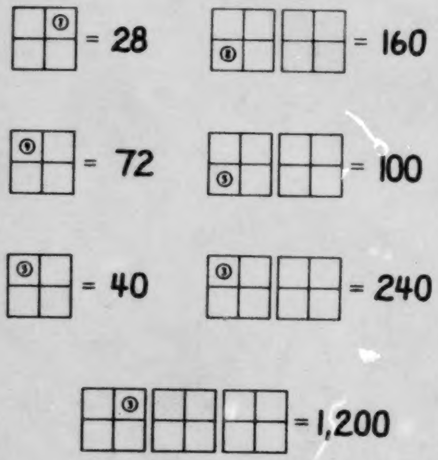
(Other colorings are possible.)

Fill in the boxes for the arrows and label the dots.


+ 7
+ 4
24
17
28
10
32
3
39
-1
43

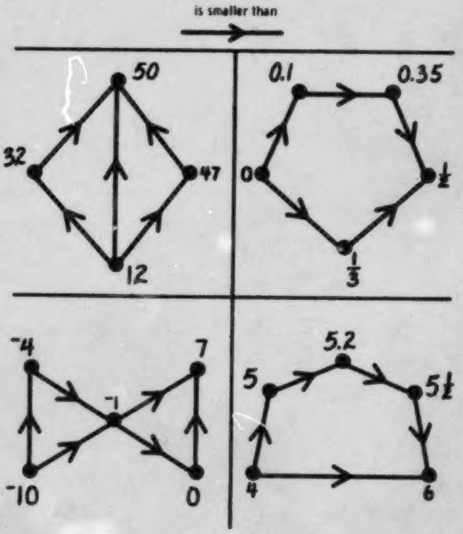
Put each number on the Minicomputer using exactly one of these checkers.

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨


⑦
= 28
⑧
= 160
⑨
= 72
⑤
= 100
⑤
= 40
③
= 240
③
= 1,200

(Other solutions are possible.)

Label the dots. Many solutions are possible.

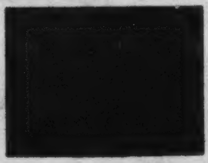

is smaller than
50
32
47
12
0.1
0.35
0
1/2
1/3
-4
7
-1
-10
0
5.2
5
5 1/2
4
6

(Many solutions are possible.)

How many squares of this size, ■ fit into the red region? 14

How many squares of this size, ■ fit into the blue region? 16

---

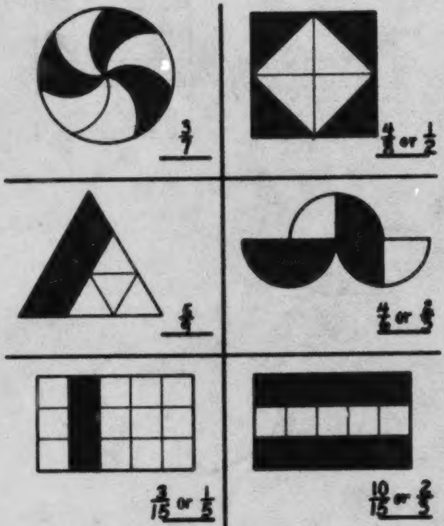


How many triangles of this size, ◢ fit into the red region? 5

How many triangles of this size, ◢ fit into the blue region? 7

---

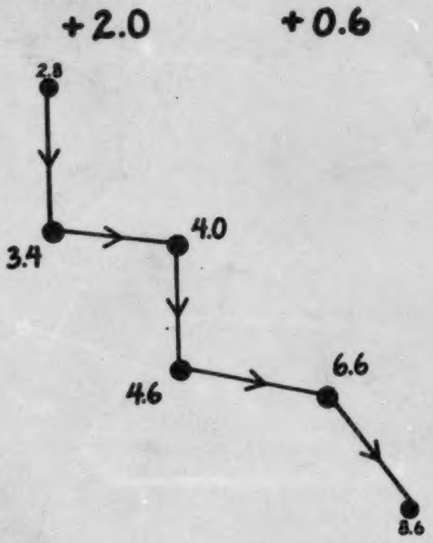
Write a fraction to indicate what part of the shape is colored blue. One is done for you.


3/7
4/8 or 1/2
5/9
4/6 or 2/3
3/15 or 1/5
10/15 or 2/3

(The fractions given here are suggested by the picture; however, other correct names could be given.)

---

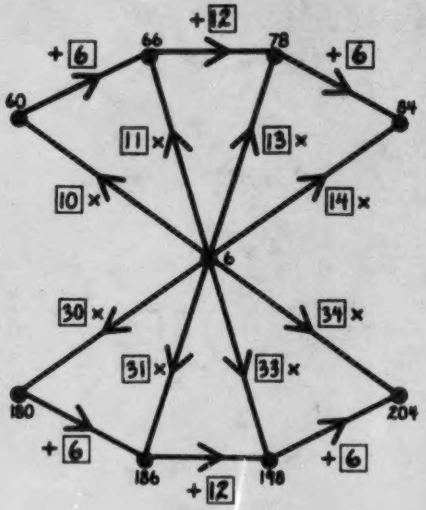
Draw a red-blue arrow road from 2.8 to 8.6.


+2.0
+0.6
2.8
3.4
4.0
4.6
6.6
8.6

(Many solutions are possible. The shortest roads have five arrows.)

---

Fill in the box for each arrow.


+12
+6
66
78
+6
60
84
11 ×
13 ×
10 ×
14 ×
6
30 ×
34 ×
31 ×
33 ×
180
204
+6
186
198
+6
+12

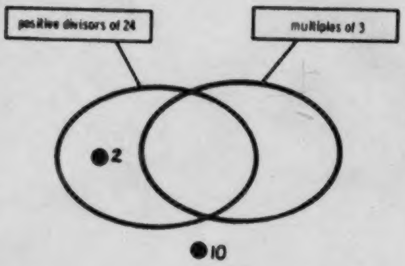
The red tag is one of these:

- larger than $^{-}10$
- even numbers
- multiples of 3
- multiples of 4
- positive divisors of 20
- positive divisors of 24

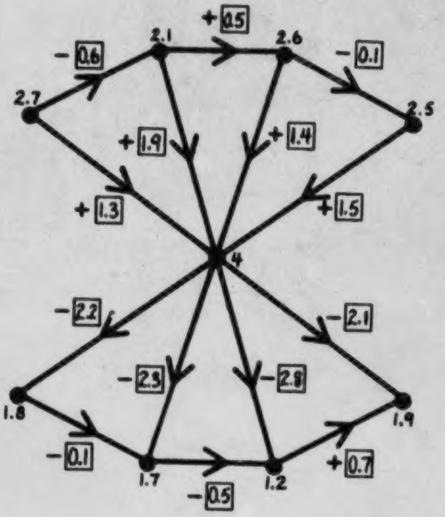
The blue tag is one of these:

- larger than 5
- even numbers
- multiples of 3
- multiples of 5
- positive divisors of 16
- positive divisors of 20

Label the strings.


positive divisors of 24
multiples of 3
2
10

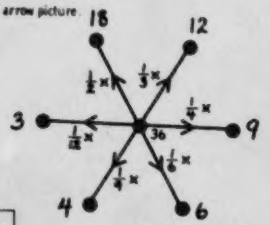
Fill in the box for each arrow.


+ 0.5
2.1
2.6
− 0.6
− 0.1
2.7
2.5
+ 1.9
+ 1.4
+ 1.3
+ 1.5
4
− 2.2
− 2.1
− 2.3
− 2.8
1.8
1.9
− 0.1
1.7
1.2
+ 0.7
− 0.5

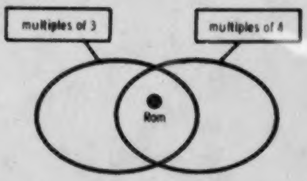
Ron is a secret number.

First clue

Ron is in this arrow picture.

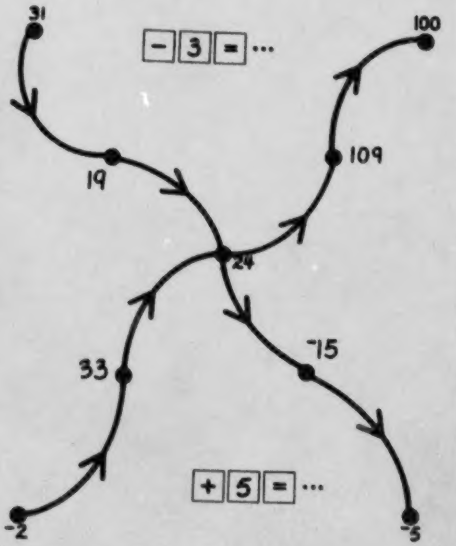

18
12
$\frac{1}{2}\times$
$\frac{1}{3}\times$
$\frac{1}{4}\times$
3
36
9
$\frac{1}{12}\times$
$\frac{1}{6}\times$
$\frac{1}{9}\times$
4
6

Second clue


multiples of 3
multiples of 4
Ron

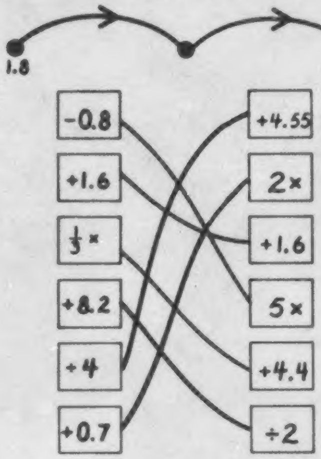
Who is Ron? 12

Label the dots. Many solutions are possible.


31
− 3 = ...
100
19
109
24
$^{-}15$
33
+ 5 = ...
$^{-}2$
$^{-}5$

(Many solutions are possible.)

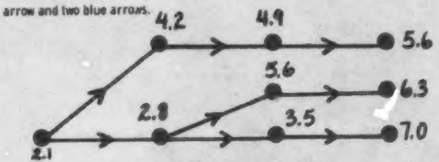
Pair the tags.


1.8
5
-0.8
+1.6
⅓×
+8.2
÷4
+0.7
+4.55
2×
+1.6
5×
+4.4
÷2

Val is a secret number.

First clue **2×** **+0.7**

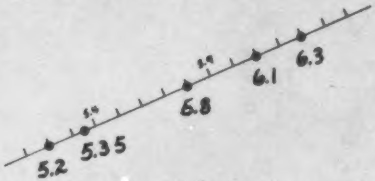
Val is the ending number of a road starting at 2.1 and using exactly one red arrow and two blue arrows.


4.2
4.9
5.6
5.6
6.3
2.8
3.5
7.0
2.1

Val could be 5.6, 6.3, or 7.0.

Second clue

Val is one of these dots. Label the dots.

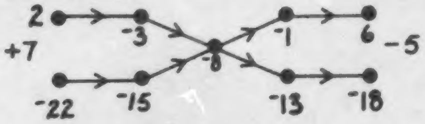

5.2
5.35
6.8
6.1
6.3

Val is 6.3.

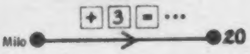
Milo is a secret number.

First clue

Milo is in this arrow picture.

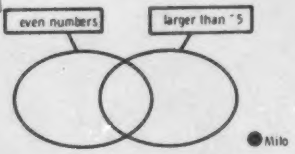

2
⁻3
+7
⁻8
⁻1
6
−5
⁻22
⁻15
⁻13
⁻18

Second clue



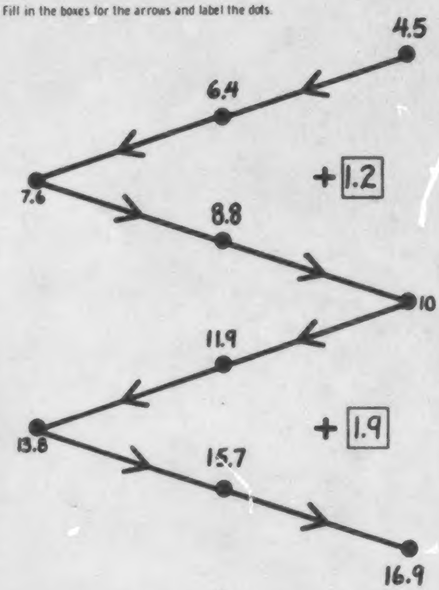
Milo could be ⁻22, ⁻13, ⁻1, or 2.

Third clue


even numbers
larger than ⁻5
Milo

Milo is ⁻13.

Fill in the boxes for the arrows and label the dots.


4.5
6.4
7.6
+ 1.2
8.8
10
11.9
+ 1.9
13.8
15.7
16.9

Put each number on the display of a hand-calculator using just these keys:

$\boxed{4}, \boxed{6}, \boxed{+}, \boxed{-}, \boxed{\times}, \boxed{\div}, \boxed{=}$

Write the keys in the order you use them. You may use a key more than once.

$\boxed{4}\boxed{\times}\boxed{6}\boxed{+}\boxed{4}\boxed{=}$ 28

$\boxed{4}\boxed{6}\boxed{+}\boxed{6}\boxed{4}\boxed{-}\boxed{4}\boxed{-}\boxed{6}\boxed{=}$ 100

$\boxed{6}\boxed{\times}\boxed{=}\boxed{+}\boxed{=}$ 72

$\boxed{6}\boxed{\div}\boxed{6}\boxed{=}$ 1

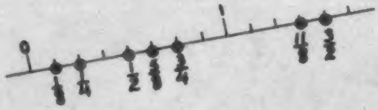
$\boxed{4}\boxed{\times}\boxed{6}\boxed{+}\boxed{6}\boxed{\div}\boxed{6}\boxed{=}$ 5

(Many solutions are possible. See Section 2 (f) of the General Introduction for a discussion of hand-calculator characteristics which could affect the solutions.)

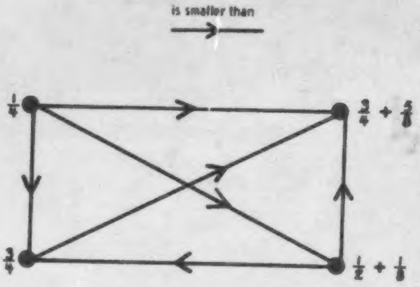
22

Locate these numbers on the number line.

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{3}{4}$ $\frac{1}{8}$ $\frac{5}{8}$ $\frac{11}{8}$ $\frac{3}{2}$



Draw all the missing red arrows between these dots.



23

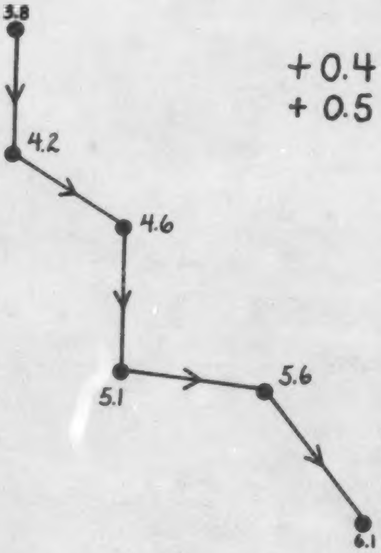
In each row, show how many of each coin is needed to make exactly $1.00. Use exactly the number of coins at the left. The first row is done for you.

| Number of coins | Half-dollar .50 | Quarter .25 | Dime .10 | Nickel .05 | Penny .01 |
|---|---|---|---|---|---|
| 3 | 1 | 2 | 0 | 0 | 0 |
| 4 | 0 | 4 | 0 | 0 | 0 |
| 6 | 1 | 0 | 5 | 0 | 0 |
|  | 1 | 1 | 1 | 3 | 0 |
|  | 0 | 3 | 2 | 1 | 0 |
| 12 | 0 | 2 | 0 | 10 | 0 |
|  | 0 | 3 | 0 | 4 | 5 |
|  | 1 | 0 | 3 | 3 | 5 |
| 20 | 0 | 0 | 0 | 20 | 0 |
|  | 0 | 0 | 8 | 2 | 10 |
|  | 0 | 3 | 0 | 2 | 15 |
| 43 | 0 | 2 | 1 | 0 | 40 |
|  | 0 | 0 | 1 | 12 | 30 |
|  | 0 | 1 | 1 | 6 | 35 |

24

(Other solutions for 12, 20, and 43 coins are possible.)

Draw a red-blue arrow road from 3.8 to 6.1.

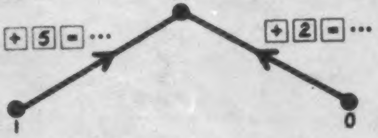


25

(Many solutions are possible. The shortest roads have five arrows.)

Mot is a secret number.

First clue



Mot could be 6, 16, 26, 36, 46, 56, 66 and so on.

Second clue

Mot is smaller than 200.

Mot =₃1

Mot could be 16, 46, 76, 106, 136, 166 or 196

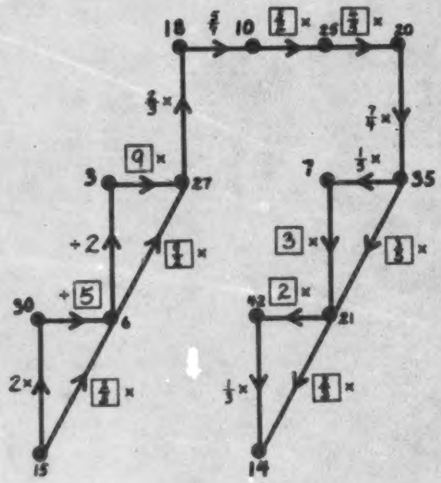
Third clue

Mot can be written by adding parentheses to this expression.

$$4 \times (8 + 9) \times 2 = \text{Mot}$$

Mot is 136.

Label the dots and fill in the boxes for the arrows.



Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

Multiply

$$\begin{array}{r} \boxed{4}\,\boxed{3}\,8\,\boxed{7} \\ \times \quad 7 \\ \hline \boxed{3}\,0\,7\,\boxed{0}\,9 \end{array}$$

Multiply

$$\begin{array}{r} \boxed{4}\,6\,9\,\boxed{7} \\ \times \quad 4 \\ \hline 1\,8\,\boxed{7}\,8\,8 \end{array}$$

Multiply

$$\begin{array}{r} 360 \\ \times\ 68 \\ \hline 2880 \\ 2160\phantom{0} \\ \hline 24{,}480 \end{array}$$

Divide

$$\begin{array}{rl} \boxed{5}\,\boxed{9}\,\boxed{7} & \text{R} = \boxed{3} \\ 6\,\big)\,3\,5\,\boxed{8}\,\boxed{5} & \\ -\boxed{3}\,\boxed{0}\,\boxed{0}\,\boxed{0} & \boxed{5}\,0\,0 \\ \hline \boxed{5}\,\boxed{8}\,\boxed{5} & \\ 5\,4\,0 & \boxed{9}\,0 \\ \hline 4\,5 & \\ \boxed{4}\,\boxed{2} & \boxed{7} \\ \hline \boxed{3} & \end{array}$$

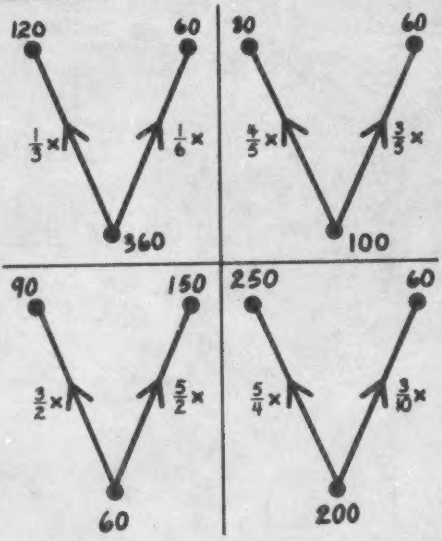
Divide

$$\begin{array}{r} 4{,}278\ \text{R}=3 \\ 9\,\big)\,38{,}505 \end{array}$$

Divide

$$\begin{array}{rl} \boxed{3}\,\boxed{5}\,\boxed{8} & \text{R} = \boxed{6} \\ 15\,\big)\,5\,3\,\boxed{7}\,\boxed{6} & \\ -\boxed{4}\,\boxed{5}\,\boxed{0}\,\boxed{0} & \boxed{3}\,0\,0 \\ \hline \boxed{8}\,\boxed{7}\,\boxed{6} & \\ -\ \boxed{7}\,\boxed{5}\,\boxed{0} & \boxed{5}\,0 \\ \hline \boxed{1}\,2\,\boxed{6} & \\ -\ \boxed{1}\,\boxed{2}\,\boxed{0} & \boxed{8} \\ \hline 6 & \end{array}$$

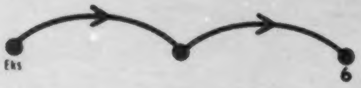
60 is the smallest number in each arrow picture.
Label the dots.



Eks is a secret number.

| The RED arrow could be for: |
|---|
| 2× |
| 10× |
| $\frac{1}{3}$× |

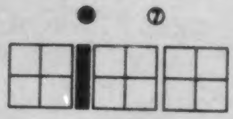
| The BLUE arrow could be for: |
|---|
| 4× |
| 10× |
| $\frac{1}{3}$× |



Eks could be 0.06, 0.15, 0.3, 0.75, 1.8, 4.5,
9, or 54.

**Second clue**

Eks can be put on the Minicomputer with exactly these checkers.



Eks could be 0.15, 0.3, 1.8, or 9.

**Third clue**

Eks can be written by adding parentheses to this expression.

$$4 \times (0.2 + (0.5 \div 2))$$

Eks is 1.8.

Put each number on the hand-calculator using just these keys:

$\boxed{2}$, $\boxed{4}$, $\boxed{6}$, $\boxed{8}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, $\boxed{\%}$, $\boxed{=}$

Write the keys in the order you use them. You may use a key more than once.

It costs 1¢ each time you press a key. Try to spend less than the amount shown for each number.

$\boxed{4}\boxed{\times}\boxed{6}\boxed{\times}\boxed{6}\boxed{+}\boxed{6}\boxed{\div}\boxed{2}\boxed{=}$ 75 [11¢]

$\boxed{2}\boxed{-}\boxed{4}\boxed{2}\boxed{=}$ ⁻40 [8¢]

$\boxed{8}\boxed{\div}\boxed{8}\boxed{-}\boxed{6}\boxed{-}\boxed{8}\boxed{=}$ ⁻13 [11¢]

$\boxed{8}\boxed{+}\boxed{6}\boxed{\div}\boxed{4}\boxed{=}$ 3.5 [10¢]

$\boxed{2}\boxed{+}\boxed{8}\boxed{\div}\boxed{=}\boxed{=}\boxed{\times}\boxed{6}\boxed{+}\boxed{2}\boxed{\div}\boxed{2}\boxed{=}$ 1.3 [15¢]

(Many solutions are possible. See Section 2 (f) of the General Introduction for a discussion of hand-calculator characteristics which could affect the solutions.)

# ARCADE
# OF
# PROBLEMS
# #2

## ARCADE OF PROBLEMS #2

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.* You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems.

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* See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

# W 3&4

## Lesson 1

Give each student a copy of "Arcade of Problems #2" and let them work on their own.

If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

## Lesson 2

Return the students' copies of "Arcade of Problems #2". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

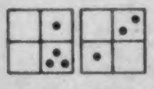
431

# Arcade of Problems # 2

Kimo is a secret number.

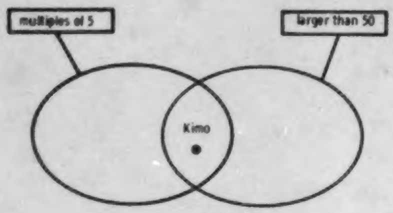
**First clue**

Kimo can be shown on this Minicomputer by taking off exactly one checker.



Kimo could be 40, 70, 76, or 78.

**Second clue**

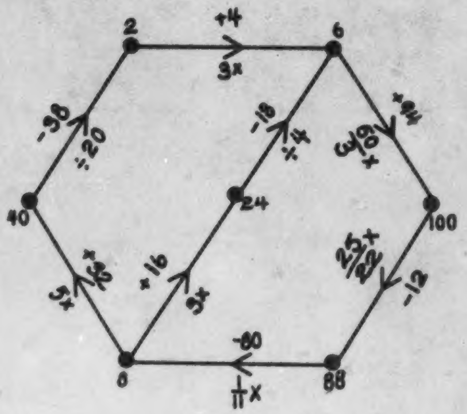

multiples of 5
larger than 50
Kimo

Kimo is 70.

2

Label the arrows; try to label some arrows in two ways. One is done for you.

Many solutions are possible.

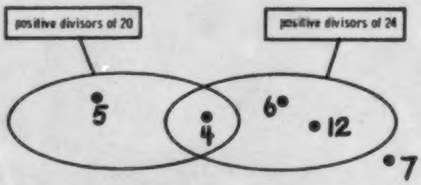

2
+4
3x
6
-38
÷20
-18
÷4
+94
50/3 x
40
24
100
+32
5x
+16
3x
25/22 x
-12
8
-80
1/11 x
88

3

(Other solutions are possible.)

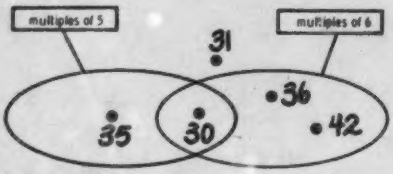
Put each of these numbers in the string picture.

4 5 6 7 12


positive divisors of 20
positive divisors of 24
5
4
6
12
7

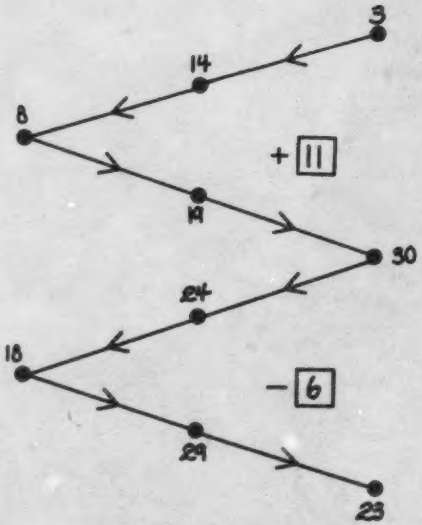
Put each of these numbers in the string picture.

30 31 35 36 42


multiples of 5
multiples of 6
31
35
30
36
42

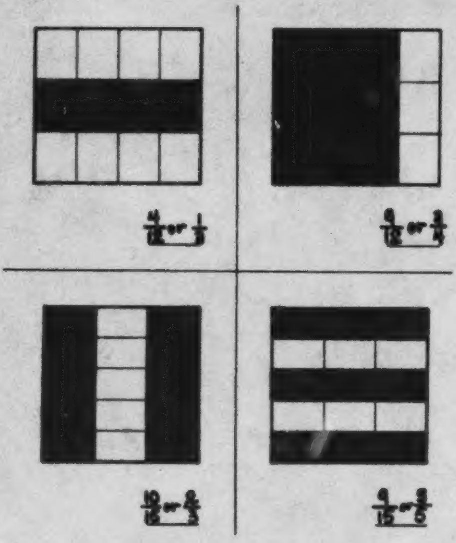
4

Fill in the boxes for the arrows and label the dots.


3
14
8
+ 11
19
30
24
18
− 6
29
23

5

Write a fraction for each picture to indicate what part of the rectangle is shaded red.

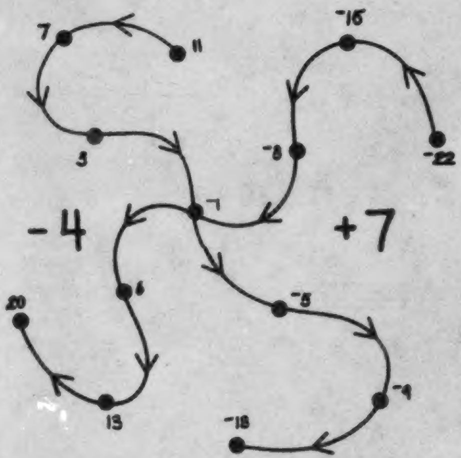


6

(The fractions given here are suggested by the pictures; however, other correct names could be given.)

---

Label the dots.



What is the smallest number in this picture? ⁻22

The only multiples of 4 in this picture are ⁻8 and 20.

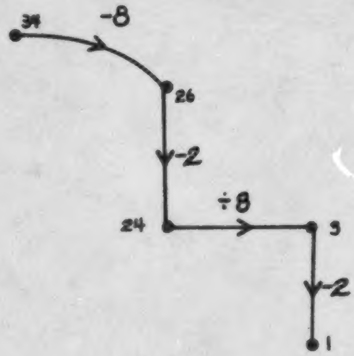
7

---

Build an arrow road from 34 to 1. Each arrow must be for one of these:

+2, +8, -2, -8, x2, x8, ÷2, ÷8

One arrow in a road is labeled for you.



8

(Many solutions are possible. The shortest roads have four arrows.)

---

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Add

$$\begin{array}{r} 1\,\boxed{9}\,8\,\boxed{3}\,2 \\ +\;5\,\boxed{6}\,3\,8 \\ \hline \boxed{2}\,5\,4\,7\,0 \end{array}$$

Add

$$\begin{array}{r} \boxed{3}\,5\,6 \\ 7\,\boxed{8} \\ +\;3\,\boxed{4}\,2 \\ \hline 7\,7\,6 \end{array}$$

Subtract

$$\begin{array}{r} \boxed{7}\,6\,3\,\boxed{9} \\ -\;2\,\boxed{4}\,9\,4 \\ \hline 5\,1\,\boxed{4}\,5 \end{array}$$

Subtract

$$\begin{array}{r} 7\,0\,4\,\boxed{1} \\ -\;4\,\boxed{3}\,3 \\ \hline \boxed{6}\,\boxed{6}\,0\,8 \end{array}$$

Subtract

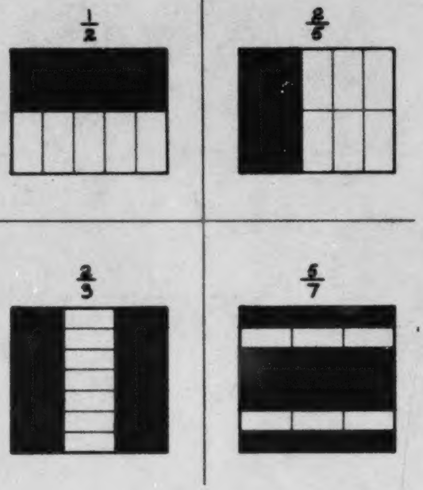
$$\begin{array}{r} 36{,}504 \\ -\;327 \\ \hline 36{,}177 \end{array}$$

Subtract

$$\begin{array}{r} 8\,0\,3\,\boxed{8} \\ -\;\boxed{6}\,7\,\boxed{6}\,6 \\ \hline 1\,\boxed{2}\,7\,2 \end{array}$$

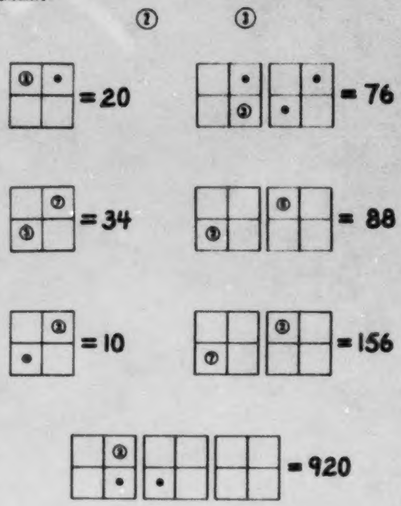
9

Color in red the indicated part of each rectangle.

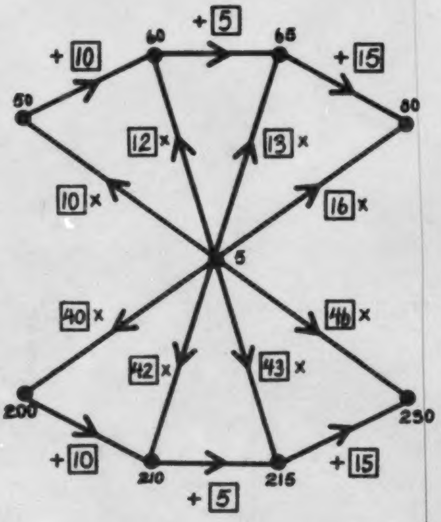

$\frac{1}{2}$
$\frac{2}{5}$
$\frac{2}{3}$
$\frac{5}{7}$

(Many solutions are possible.)

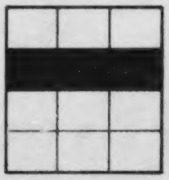
Put each number on the Minicomputer by adding exactly one of these checkers.


①
③
= 20
= 76
= 34
= 88
= 10
= 156
= 920

Fill in the boxes for the arrows.

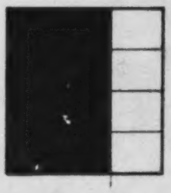

+ 10
60
+ 5
65
+ 15
50
80
12 x
13 x
10 x
16 x
5
40 x
46 x
42 x
43 x
200
250
+ 10
210
215
+ 15
+ 5

Color one-fourth of this region red and fill in the box.



$$\frac{1}{4} = \frac{\boxed{3}}{12}$$

Color two-thirds of this region red and fill in the box.



$$\frac{2}{3} = \frac{\boxed{8}}{12}$$

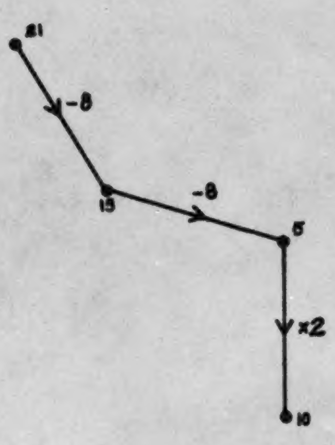
Complete.

$$\frac{1}{4} + \frac{2}{3} = \frac{\boxed{3}}{12} + \frac{\boxed{8}}{12} = \frac{11}{12}$$

(Other colorings are possible.)

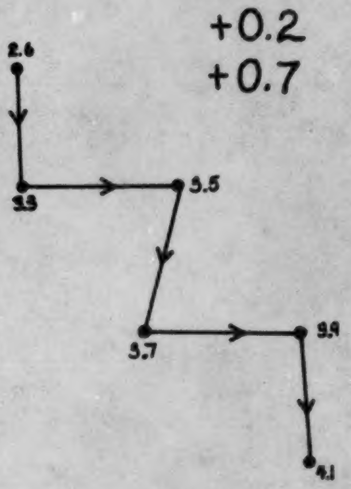
Build an arrow road from 21 to 10. Each arrow must be for one of these:

+2, +8, -2, -8, x2, x8, ÷2, ÷8


21
-8
13
-8
5
x2
10

(Many solutions are possible. The shortest roads have three arrows.)

Build a red-blue road from 2.6 to 4.1.


+0.2
+0.7
2.6
3.3
3.5
3.7
3.9
4.1

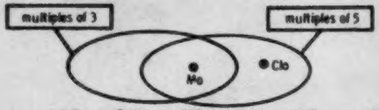
(The road must have one blue and three red arrows, but they may be in a different order.)

Clo and Mo are secret whole numbers.

First clue

Clo is between 0 and 50.
Mo is between 0 and 100.

Second clue


multiples of 3
multiples of 5
Mo
Clo

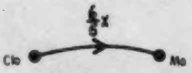
Mo could be 15, 30, 45, 60, 75, or 90.
Clo could be 5, 10, 20, 25, 35, or 40.

Third clue

Mo =₄ 2        Clo =₃ 1

Mo could be 30 or 90.
Clo could be 10, 25, or 40.

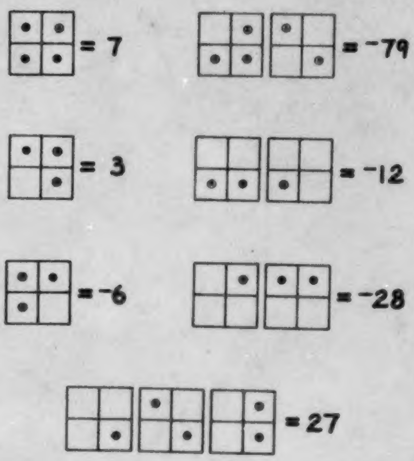
Fourth clue



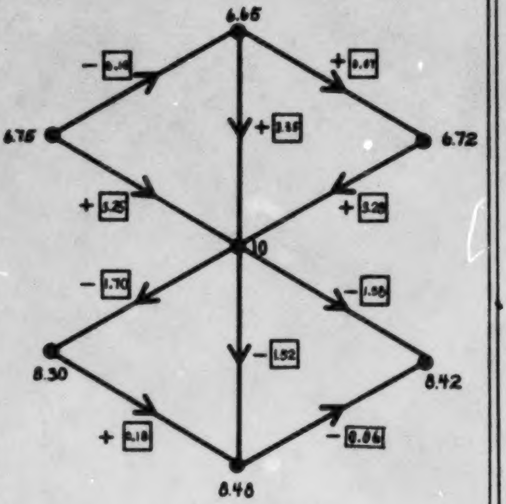
Who is Clo? 25
Who is Mo? 30

Put each number on the Minicomputer by adding exactly one of these checkers.

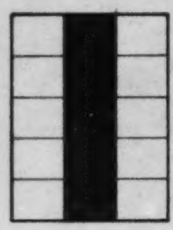
⊘ ●


= 7
= −79
= 3
= −12
= −6
= −28
= 27

Fill in the boxes for the arrows.

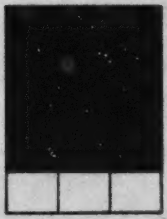

6.65
6.75
6.72
10
8.30
8.42
8.48
− 0.10
+ 0.07
+ 3.35
+ 3.25
+ 3.28
− 1.70
− 1.58
− 1.52
+ 0.18
− 0.06

Color one-third of this region red and fill in the box.



$$\frac{1}{3} = \frac{\boxed{5}}{15}$$

Color four-fifths of this region red and fill in the box.



$$\frac{4}{5} = \frac{\boxed{12}}{15}$$

Complete.

$$\frac{1}{3} + \frac{4}{5} = \underline{\frac{17}{15} = 1\frac{2}{15}}$$

(Other colorings are possible.)

Fill in the boxes and write a decimal name for each number. The first one is done for you.

$$\frac{1}{5} = \frac{\boxed{2}}{10} = \underline{0.2}$$

$$\frac{1}{2} = \frac{\boxed{5}}{10} = \underline{0.5}$$

$$\frac{3}{5} = \frac{\boxed{6}}{10} = \underline{0.6}$$

$$\frac{4}{5} = \frac{\boxed{8}}{10} = \underline{0.8}$$

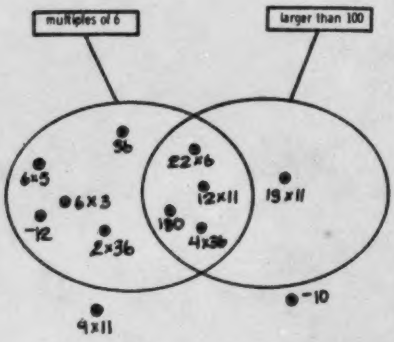
$$\frac{1}{4} = \frac{\boxed{25}}{100} = \underline{0.25}$$

$$\frac{3}{4} = \frac{\boxed{75}}{100} = \underline{0.75}$$

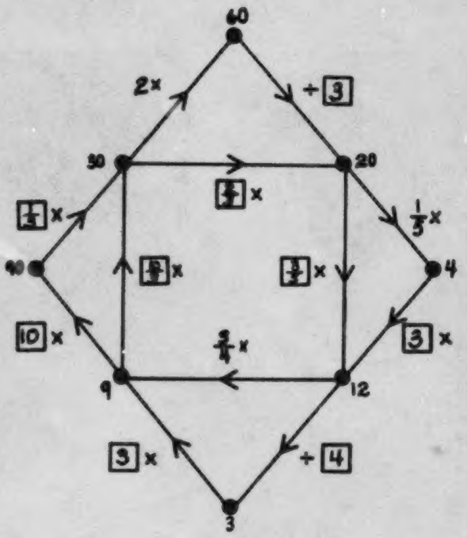
$$\frac{1}{8} = \frac{\boxed{125}}{1{,}000} = \underline{0.125}$$

Put these numbers in the string picture.

| | | | |
|---|---|---|---|
| 6 × 5 | 9 × 11 | −10 | 36 |
| 6 × 3 | 12 × 11 | −12 | 2 × 36 |
| 22 × 6 | 13 × 11 | 180 | 4 × 36 |

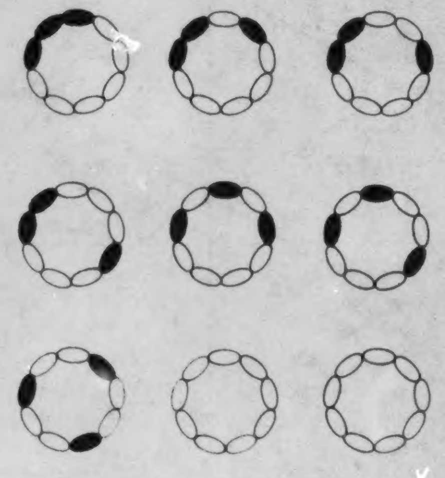

multiples of 6
larger than 100
36
6×5
6×3
−12
2×36
22×6
12×11
180
4×36
19×11
−10
9×11

Label the dots and fill in the boxes for the arrows.

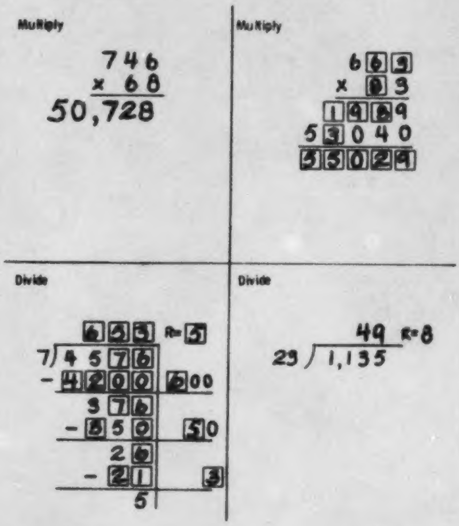

60
2×
÷ 3
30
20
1/3 ×
40
4
10 ×
3 ×
3/4 ×
9
12
3 ×
÷ 4
3

Show all of the different necklaces with six white and three red beads.

You may not need to use all of the necklaces drawn.



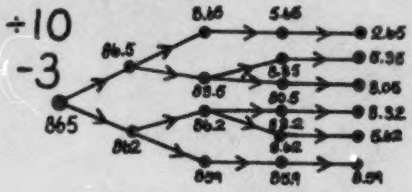
Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.


Multiply
746
× 68
50,728
Multiply
669
× 83
1989
53040
55029
Divide
653 R=5
7) 4576
− 4200 600
376
− 350 50
26
− 21 3
5
Divide
49 R=8
23 ) 1,135

Miff is a secret number.

First clue

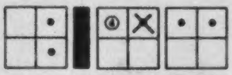
Miff is the ending number of an arrow road starting at 865 and using exactly two red arrows and two blue arrows.


÷10
−3
865
86.5
862
8.65
5.65
83.5
86.2
859
2.65
8.35
8.05
8.32
5.62
8.59

Miff could be 2.65, 6.35, 5.62, 8.05, 8.32, or 8.59.

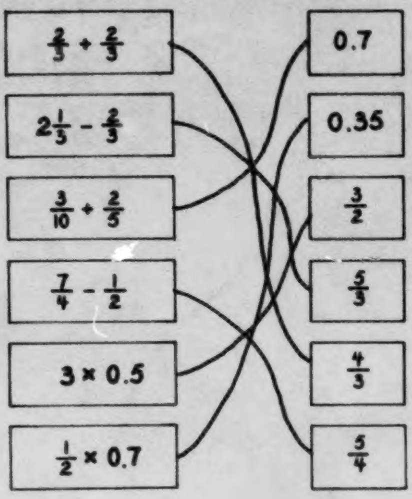
Second clue

Miff can be shown on this Minicomputer by taking off exactly one checker.



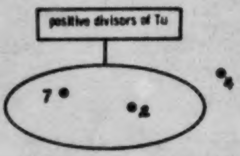
Who is Miff? 8.32

Pair the tags.


$\frac{2}{3} + \frac{2}{3}$
$2\frac{1}{3} - \frac{2}{3}$
$\frac{3}{10} + \frac{2}{5}$
$\frac{7}{4} - \frac{1}{2}$
3 × 0.5
$\frac{1}{2}$ × 0.7
0.7
0.35
$\frac{3}{2}$
$\frac{5}{3}$
$\frac{4}{3}$
$\frac{5}{4}$

Tu is a secret whole number smaller than 100.

First clue


positive divisors of Tu
7
2
4

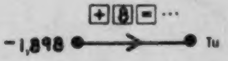
Tu could be 14 , 42 , 70 , or 98 .

Second clue

$Tu =_3 2$

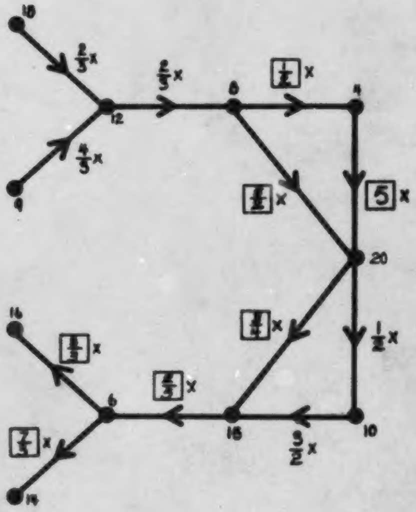
Tu could be 14 or 98 .

Third clue

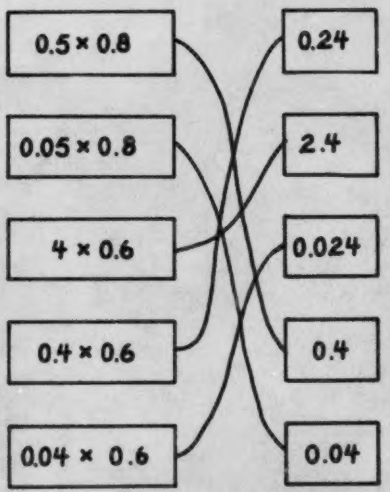


Who is Tu? 14

Label the dots and fill in the boxes for the arrows.


18
$\frac{2}{3}$x
9
$\frac{4}{3}$x
12
$\frac{2}{3}$x
8
$\frac{1}{2}$x
4
$\frac{5}{2}$x
5x
20
$\frac{3}{4}$x
$\frac{1}{2}$x
10
$\frac{3}{2}$x
15
$\frac{2}{5}$x
6
$\frac{8}{3}$x
16
$\frac{7}{3}$x
14

Pair the tags.


0.5 × 0.8
0.05 × 0.8
4 × 0.6
0.4 × 0.6
0.04 × 0.6
0.24
2.4
0.024
0.4
0.04

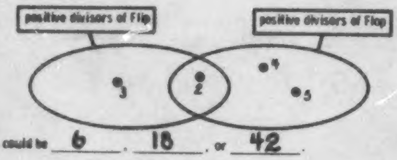
Flip and Flop are secret whole numbers.

**First clue**

Flip is smaller than 50.

Flop is smaller than 100.

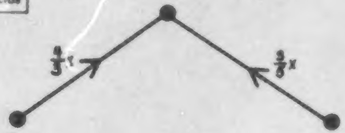
**Second clue**


positive divisors of Flip
positive divisors of Flop
3
2
4
5

Flip could be 6, 18, or 42.

Flop could be 20, 40, or 80.

**Third clue**

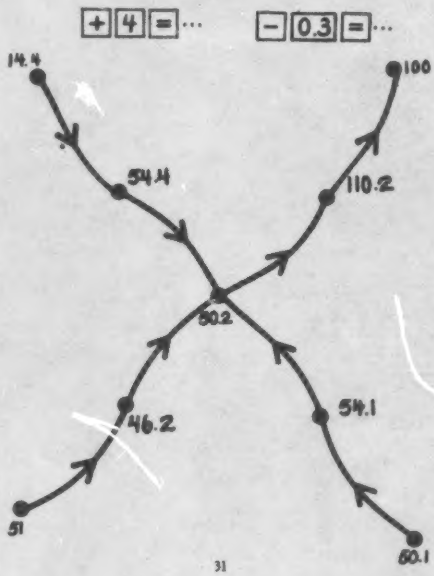

$\frac{4}{3}x$
$\frac{3}{5}x$
Flip
Flop

Who is Flip? 18

Who is Flop? 40

---

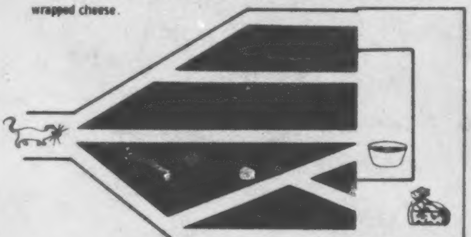
Label the dots. Many solutions are possible.


+ 4 = ...
− 0.3 = ...
14.4
54.4
50.2
46.2
51
100
110.2
54.1
50.1

(Many solutions are possible.)

---

A mouse enters the maze drawn below. Assume that the mouse never turns back and that it randomly decides which path to follow. It cannot smell the wrapped cheese.



What are the mouse's chances of finding : Cheese? $\frac{5}{12}$ (5 chances out of 12)

Water? $\frac{7}{12}$ (7 chances out of 12)

You may use this square or any other method to solve this problem.

Red : cheese
Blue : water

(The partition of the square may be rearranged.)

# ARCADE OF PROBLEMS

## #3

## ARCADE OF PROBLEMS #3

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.* You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems.

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* See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

# W 5&6

## Lesson 1

Give each student a copy of "Arcade of Problems #3" and let them work on their own.

If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

## Lesson 2

Return the students' copies of "Arcade of Problems #3". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

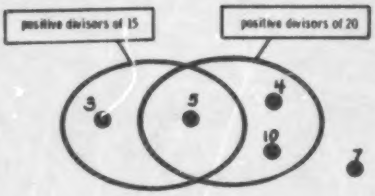
If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during free study time or as a homework assignment.

# Arcade of Problems # 3

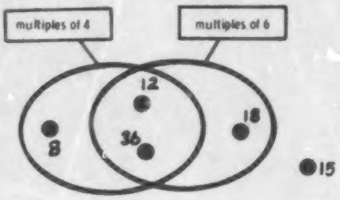
Put each of these numbers in the string picture.

3 4 5 7 10


positive divisors of 15
positive divisors of 20
3
5
4
10
7

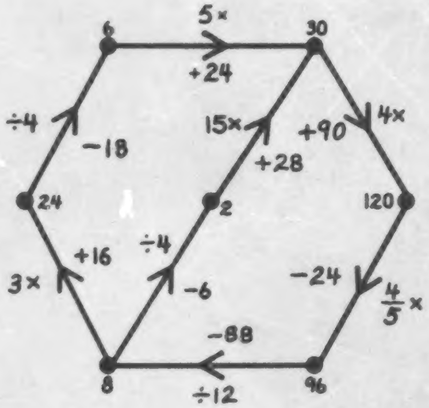
Put each of these numbers in the string picture.

8 12 15 18 36


multiples of 4
multiples of 6
12
8
36
18
15

2

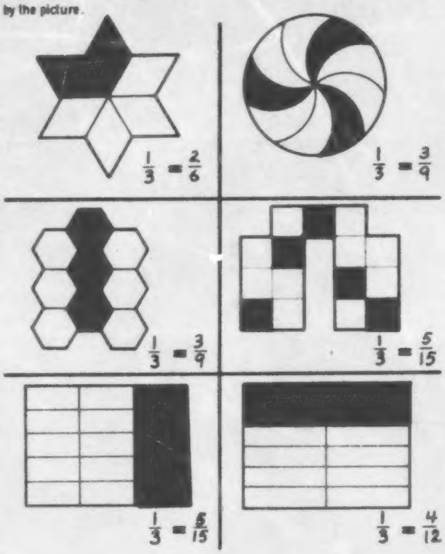
Label the arrows; try to label some arrows in two ways. One is done for you. Many solutions are possible.


6
30
5×
+24
÷4
−18
15×
+28
+90
4×
24
2
120
3×
+16
÷4
−6
−24
4/5 ×
−88
8
÷12
96

3

(Other solutions are possible.)

Color one-third of each shape blue. Write another name for $\frac{1}{3}$ as suggested by the picture.


1/3 = 2/6
1/3 = 3/9
1/3 = 3/9
1/3 = 5/15
1/3 = 5/15
1/3 = 4/12

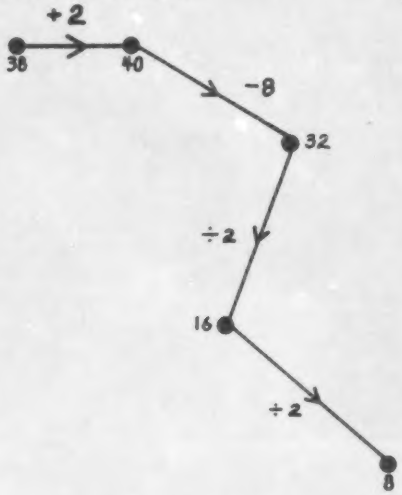
4

(Other colorings are possible.)

Build an arrow road from 38 to 8. Each arrow must be for one of these:

+2, +8, −2, −8, x2, x8, ÷2, ÷8

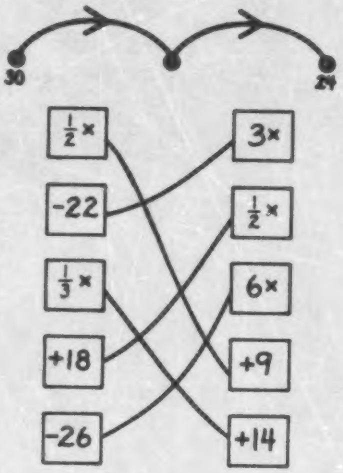
One arrow in a road is labeled for you.


+2
38
40
−8
32
÷2
16
÷2
8

5

(Many solutions are possible.
The shortest roads have three
arrows from 40 to 8.)

Pair the tags.



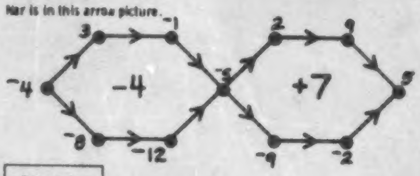
30
24
$\frac{1}{2}$×
-22
$\frac{1}{3}$×
+18
-26
3×
$\frac{1}{2}$×
6×
+9
+14

---

Nar is a secret number.

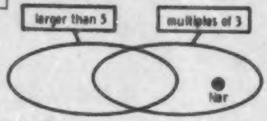
First clue

Nar is in this arrow picture.



3
⁻1
⁻4
−4
⁻8
⁻12
⁻5
2
9
+7
5
⁻9
⁻2

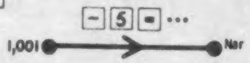
Second clue



larger than 5
multiples of 3
Nar

Nar could be 3, ⁻9, or ⁻12.

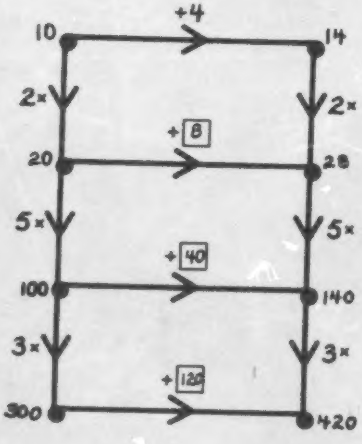
Third clue



Nar is ⁻9.

---

Label the dots and fill in the boxes for the arrows.



+4
10
14
2×
2×
+8
20
28
5×
5×
+40
100
140
3×
3×
+120
300
420

---

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Add

$$\begin{array}{r} 5\boxed{3}27 \\ +\ 7\boxed{5}9 \\ \hline \boxed{6}08\boxed{6} \end{array}$$

Add

$$\begin{array}{r} 6\boxed{8} \\ 582 \\ +\ 1\boxed{3}7 \\ \hline \boxed{7}87 \end{array}$$

Add

$$\begin{array}{r} 136 \\ 4{,}078 \\ +\ 56 \\ \hline 4{,}270 \end{array}$$

Subtract

$$\begin{array}{r} 62\boxed{8}7 \\ -\ \boxed{4}93\boxed{3} \\ \hline 1\boxed{3}54 \end{array}$$

Subtract

$$\begin{array}{r} 2{,}059 \\ -\ 1{,}673 \\ \hline 386 \end{array}$$

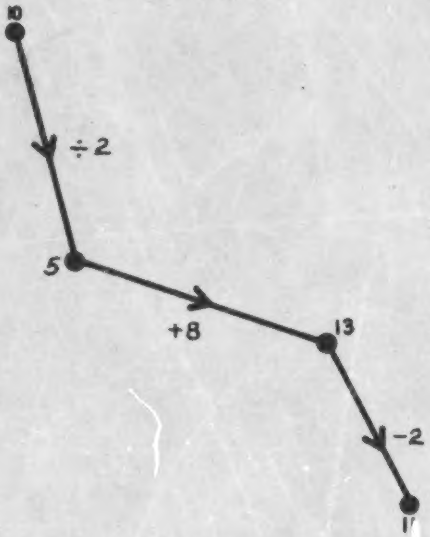
Subtract

$$\begin{array}{r} 4\boxed{3}7\boxed{0}4 \\ -\ \boxed{2}2\boxed{0} \\ \hline \boxed{4}3484 \end{array}$$

444

Build an arrow road from 10 to 11. Each arrow must be for one of these:

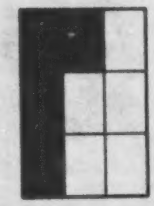
+2, +8, -2, -8, x2, x8, ÷2, ÷8


10
÷2
5
+8
13
-2
11

10

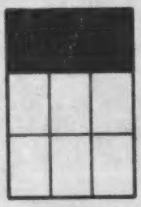
(Many solutions are possible. The shortest roads have three arrows.)

Color four-ninths of this region red.



$\frac{4}{9}$

Color one-third of this region red and fill in the box.



$\frac{1}{3} = \frac{\boxed{3}}{9}$

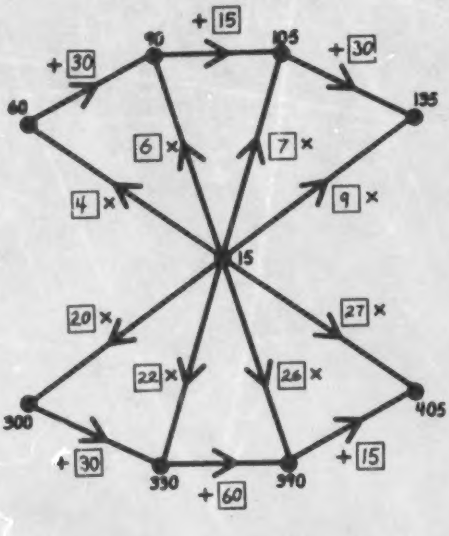
Complete.

$$\frac{4}{9} + \frac{1}{3} = \frac{4}{9} + \frac{\boxed{3}}{9} = \frac{7}{9}$$

11

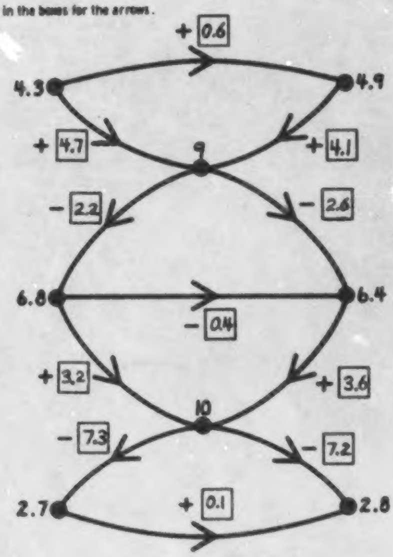
(Other colorings are possible.)

Fill in the boxes for the arrows.


+ 30
90
+ 15
105
+ 30
60
135
6 ×
7 ×
4 ×
9 ×
15
20 ×
27 ×
22 ×
26 ×
300
405
+ 30
330
+ 60
390
+ 15

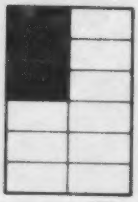
12

Fill in the boxes for the arrows.


+ 0.6
4.3
4.9
+ 4.7
+ 4.1
9
- 2.2
- 2.6
6.8
6.4
- 0.4
+ 3.2
+ 3.6
10
- 7.3
- 7.2
2.7
+ 0.1
2.8

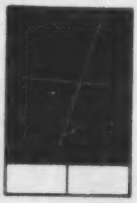
13

Color one-fourth of this region red and fill in the box.



$\frac{1}{4} = \frac{\boxed{3}}{12}$

Color five-sixths of this region red and fill in the box.



$\frac{5}{6} = \frac{\boxed{10}}{12}$

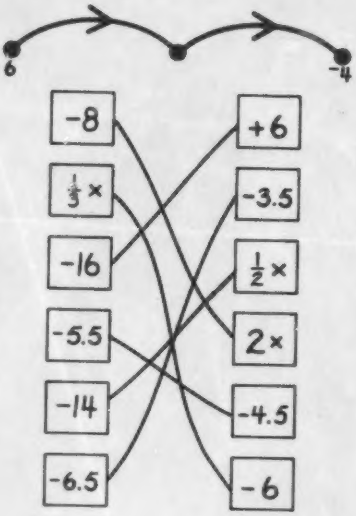
Complete.

$\frac{1}{4} + \frac{5}{6} = \frac{13}{12} = 1\frac{1}{12}$

(Other colorings are possible.)

---

Pair the tags.


6
-4
-8
$\frac{1}{3}$×
-16
-5.5
-14
-6.5
+6
-3.5
$\frac{1}{2}$×
2×
-4.5
-6

---

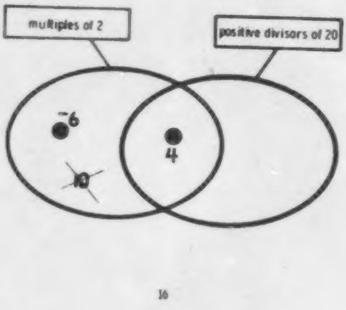
The red tag is one of these.

- multiples of 2
- multiples of 3
- multiples of 4
- positive divisors of 12
- positive divisors of 18
- positive divisors of 20
- positive divisors of 24

The blue tag is one of these.

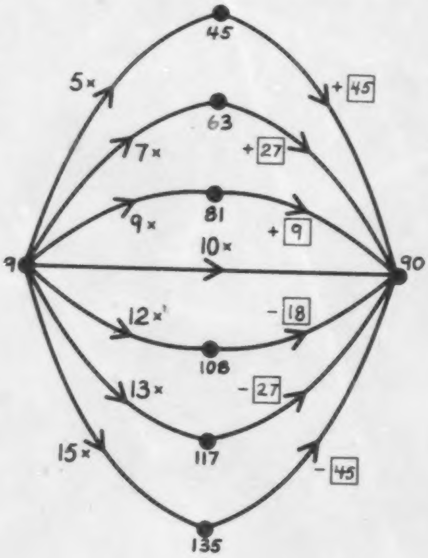
- multiples of 2
- multiples of 3
- multiples of 4
- positive divisors of 12
- positive divisors of 18
- positive divisors of 20
- positive divisors of 24

Label the strings.


multiples of 2
positive divisors of 20
-6
4
~~10~~

---

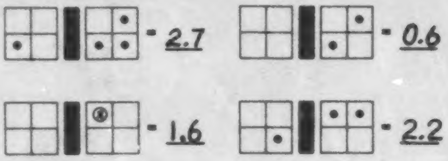
Label the dots and fill in the boxes for the arrows.


45
5×
+45
63
7×
+27
81
9×
+9
10×
9
90
12×
-18
108
13×
-27
117
15×
-45
135

Rio is a secret number.

First clue

Rio is one of these numbers.


= 2.7

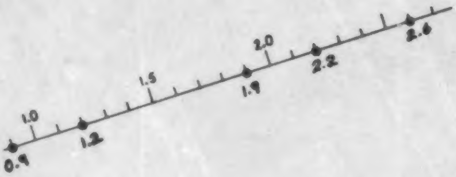
= 0.6

= 1.6

= 2.2

Second clue

Rio is one of these dots. Label the dots.


1.0
1.5
2.0
0.9
1.2
1.9
2.2
2.6

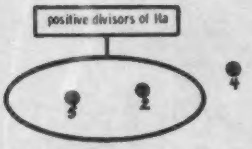
Rio is 2.2.

Ita is a secret whole number.

First clue

Ita is smaller than 100.

Second clue


positive divisors of Ita
3
2
4

Ita could be 10, 30, 50, 70, or 90.

Third clue

$\text{Ita} =_3 1$

Ita could be 10 or 70.

Fourth clue

$\text{Ita} \div 2 =_4 3$

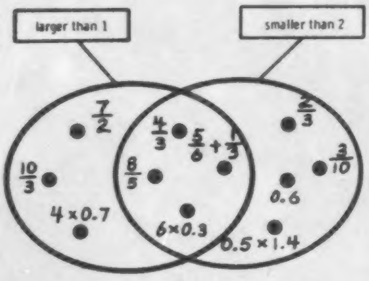
Ita is 70.

Locate these numbers in the string picture.

$\frac{7}{2}$ $\frac{2}{3}$ $\frac{4}{3}$ $\frac{10}{3}$ $\frac{8}{5}$

$\frac{3}{10}$ $0.6$ $4 \times 0.7$ $6 \times 0.3$

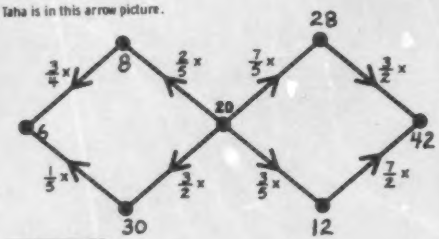
$\frac{5}{6} + \frac{1}{3}$ $0.5 \times 1.4$


larger than 1
smaller than 2
7/2
10/3
4×0.7
4/3
5/6 + 1/3
8/5
6×0.3
2/3
3/10
0.6
0.5×1.4

Taha is a secret number.

First clue

Taha is in this arrow picture.


28
8
20
6
42
30
12
3/4 x
2/5 x
7/5 x
3/2 x
1/5 x
3/2 x
3/5 x
7/2 x

Second clue

Taha is one of these numbers.

| ⊓: Greatest common divisor | ⊔: Least common multiple |
|---|---|
| $12 \sqcap 28 = 4$ | $15 \sqcup 6 = 30$ |
| $10 \sqcap 21 = 1$ | $10 \sqcup 8 = 40$ |

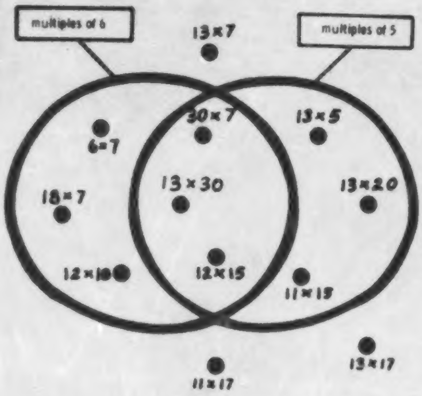
Who is Taha? 30

Locate these numbers in the string picture. One is done for you.

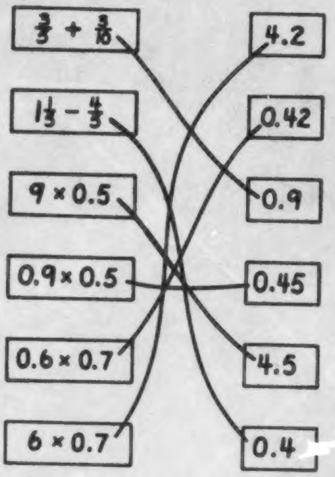
6 × 7
13 × 7
18 × 7
30 × 7

13 × 5
13 × 20
13 × 17
13 × 30

12 × 18
12 × 15
11 × 15
11 × 17


multiples of 6
13×7
multiples of 5
30×7
13×5
6×7
18×7
13×30
13×20
12×18
12×15
11×15
13×17
11×17

Pair the tags.


$\frac{3}{5}+\frac{3}{10}$
$1\frac{1}{5}-\frac{4}{5}$
9 × 0.5
0.9 × 0.5
0.6 × 0.7
6 × 0.7
4.2
0.42
0.9
0.45
4.5
0.4

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box so that the resulting number is a multiple of 9.

5$\boxed{4}$4,302

2,37$\boxed{1}$,428

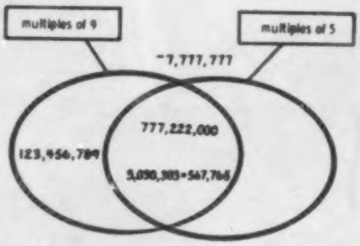
87$\boxed{8}$,769

42,307,8$\boxed{4}$8

---

Put each of these numbers in the string picture.

123,456,789
777,222,000
⁻7,777,777
3,030,303×567,765


multiples of 9
multiples of 5
⁻7,777,777
123,456,789
777,222,000
3,030,303×567,765

Mu is a secret number.

**First clue**

Mu can be written by adding parentheses to this expression.

$$8 \times 7 - 4 + 6$$

Mu could be 58, 46, 30, 72, or ⁻24.

**Second clue**

Mu is one of these numbers.

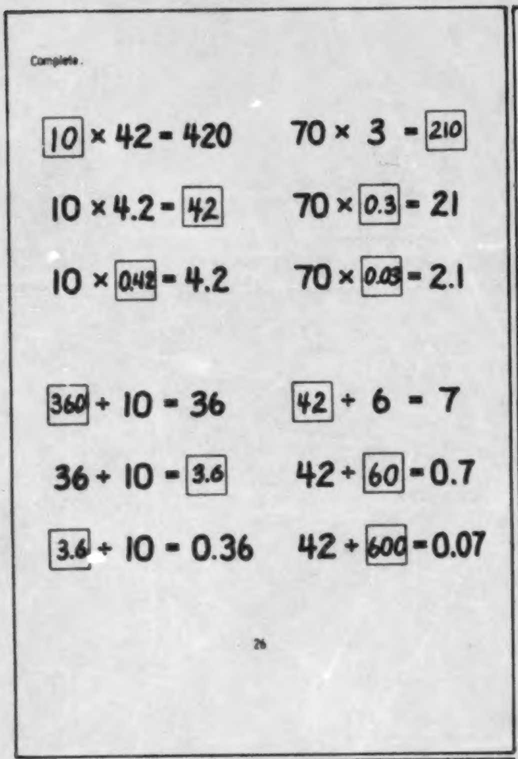
$\sqcap$: Greatest common divisor

$20 \sqcap 12 =$ 4
$30 \sqcap 42 =$ 6
$12 \sqcap 35 =$ 1

$\sqcup$: Least common multiple

$6 \sqcup 10 =$ 30
$12 \sqcup 4 =$ 12
$5 \sqcup 8 =$ 40

Who is Mu? 30



Complete.

$\boxed{10} \times 42 = 420$ $70 \times 3 = \boxed{210}$

$10 \times 4.2 = \boxed{42}$ $70 \times \boxed{0.3} = 21$

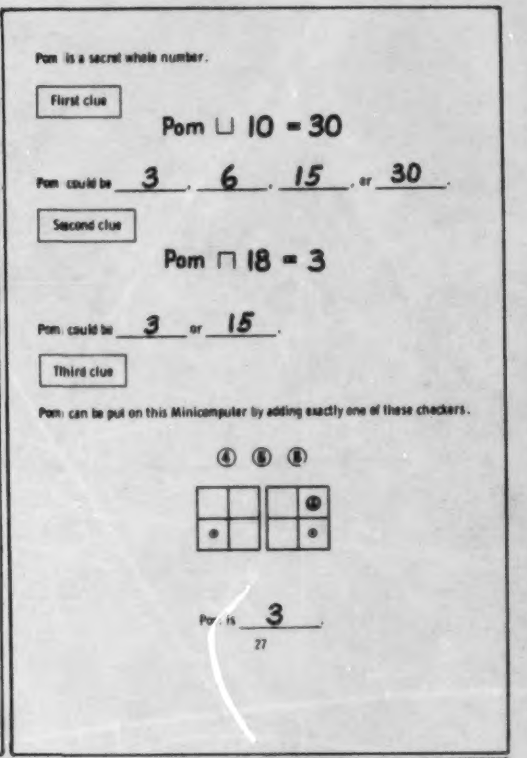
$10 \times \boxed{0.42} = 4.2$ $70 \times \boxed{0.03} = 2.1$

$\boxed{360} \div 10 = 36$ $\boxed{42} \div 6 = 7$

$36 \div 10 = \boxed{3.6}$ $42 \div \boxed{60} = 0.7$

$\boxed{3.6} \div 10 = 0.36$ $42 \div \boxed{600} = 0.07$

26



Pom is a secret whole number.

First clue

$\text{Pom} \sqcup 10 = 30$

Pom could be 3, 6, 15, or 30.

Second clue

$\text{Pom} \sqcap 18 = 3$

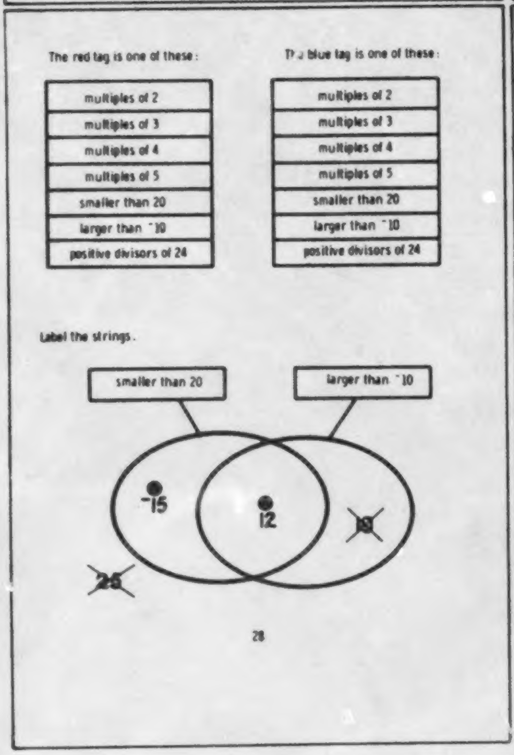
Pom could be 3 or 15.

Third clue

Pom can be put on this Minicomputer by adding exactly one of these checkers.

Pom is 3.

27



The red tag is one of these:

- multiples of 2
- multiples of 3
- multiples of 4
- multiples of 5
- smaller than 20
- larger than ⁻10
- positive divisors of 24

The blue tag is one of these:

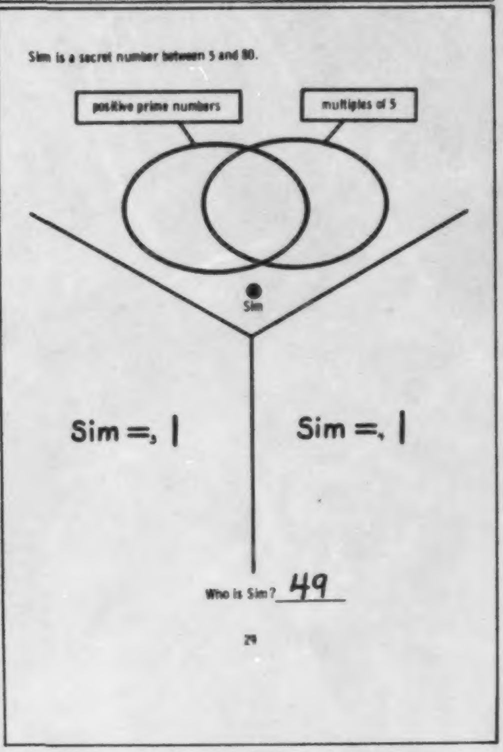
- multiples of 2
- multiples of 3
- multiples of 4
- multiples of 5
- smaller than 20
- larger than ⁻10
- positive divisors of 24

Label the strings.

smaller than 20

larger than ⁻10

28



Sim is a secret number between 5 and 80.

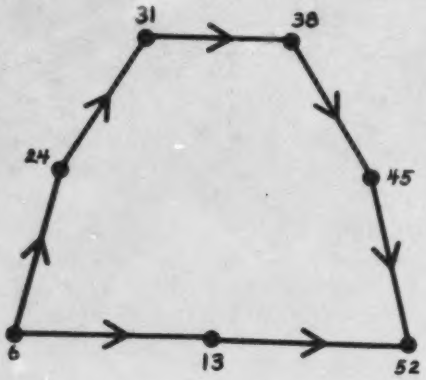
Sim =, |

Sim =, |

Who is Sim? 49

29

Label the dots and fill in the box for the red arrows.



+7 $\boxed{4}$ ×

---

Bak is a secret number.

First clue

Bak is a multiple of 9.

Second clue

Bak's name can be completed by putting one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each shape. A digit may be used several times in Bak's name.

Bak = 64 □, 34 △

Bak could be 640,341 . 641,340 . 641,349 .
642,348 . 643,347 . 644,346 . 645,345 .
646,344 . 647,343 . 648,342 . or 649,341 .

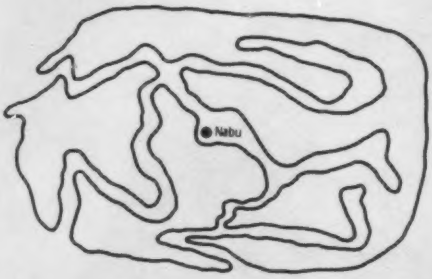
Third clue

Bak × 726,583 = 468,170,849,718

Who is Bak? 644,346

---

This picture is the map of the tunnels in a large cave. Nabu does not have a map and is lost near the center of the cave. Being lost, Nabu randomly decides which tunnels to follow. On the first try, Nabu may be lucky and find an exit or Nabu may find a dead-end and have to try again.

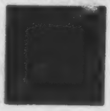


On his first try, what are Nabu's chances of finding:

a) one of the exits? $\frac{11}{24}$ (11 chances out of 24)

b) a dead-end? $\frac{13}{24}$ (13 chances out of 24)

You may use this square or any other method to solve this problem.



Red: exit
Blue: dead-end

(The partition of the square may be rearranged.)

# ELECTION
# IN THE
# NUMBER WORLD

# ELECTION IN THE NUMBER WORLD

The storybook ELECTION IN THE NUMBER WORLD provides an introduction to the full decimal positional system of numeration. Special attention is given to the relationship between the fractional and the decimal names for numbers.

The first lesson on positional systems does not make use of the storybook, but rather discusses a variety of positional notations for numbers. For example, a Base Five and a Base Twelve system are explored and in Base Twelve the students find that they must invent new names for ten and eleven.

The second lesson makes use of the storybook to study the decimal positional system of numeration.

The third lesson on the "Two-Sixteen Code" introduces a new binary-base sixteen Minicomputer code (instead of the usual binary-decimal code). It explores a Base Sixteen system using the new Minicomputer code.

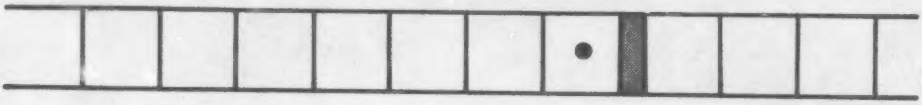
# W 7

## Lesson 1: Positional Systems

NOTE: Because there are several bases referred to in this lesson, it is necessary to be very careful in how you read numerals. For this reason, we often indicate how to read a numeral.

### Exercise 1

Draw this abacus on the chalkboard or use Minicomputer boards.[*] Put (draw) one checker on the ones' board.

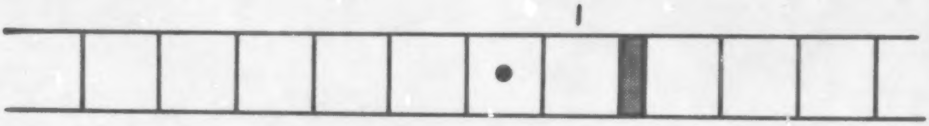


T: What number is on this abacus?

S: 1.

Label the ones' board "1".

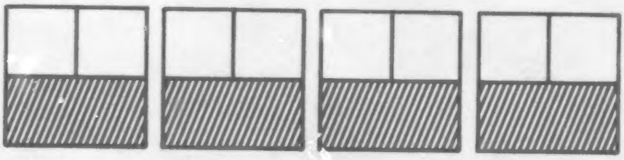
Move the checker one board to the left.



T: What number could this be?

---

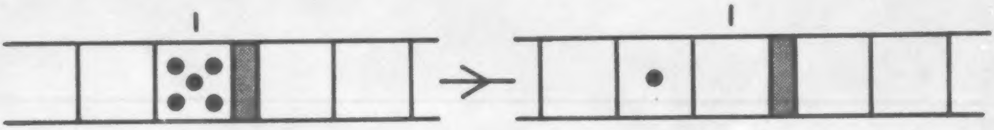
[*] If your chalkboard is not magnetic, a magnetic abacus can be constructed using your demonstration Minicomputer boards. Cover the red and the white squares with paper and use only the brown and the purple squares on each board.



The choice is arbitrary. Let the students suggest many whole numbers larger than 1. In fact, it could even be a negative integer, but do not mention this if no one suggests it.

T: But you won't know for sure what number it is until I tell you the rule for this abacus. The rule is that five checkers on a board is the same as one checker on the next board to the left.

Demonstrate the rule by putting five checkers on the ones' board and by making the following play.

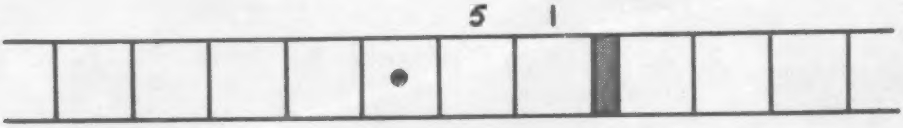


Leave the checker on the abacus.

T: Therefore, according to the rule, what number is this?

S: 5.

Label the appropriate board "5" and move the checker one board to the left.



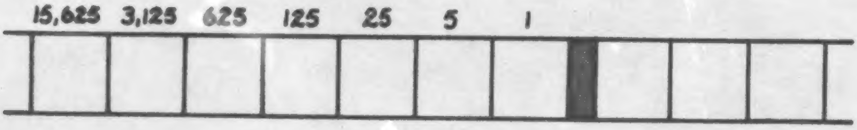
T: What number is this?

S: 25, because 5 x 5 = 25.

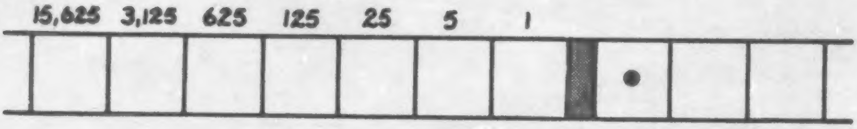
S: 25, because according to the rule we can trade this checker for five checkers on the fives' board.

# W 7

Move the checker to the left one board at a time as you ask for the value of each board. Label each board appropriately.


15,625
3,125
625
125
25
5
1

Put one checker on the abacus as shown here.


15,625
3,125
625
125
25
5
1

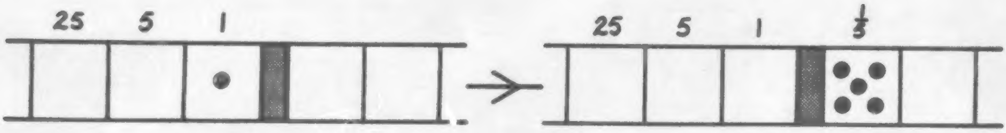
T: What number is this?

S: $\frac{1}{5}$.

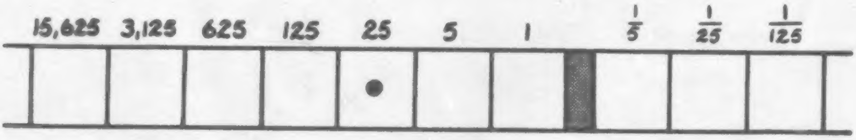
T: Why?

S: Five checkers on that board are the same as one checker on the ones' board. So the value of one checker on that board is $\frac{1}{5}$.

Show this play on the abacus.

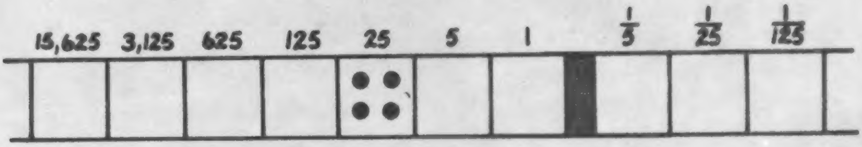

25
5
1
25
5
1
1/5

Move the checker to the right one board at a time as you ask for the value of each board. Label each board appropriately.


15,625
3,125
625
125
25
5
1
1/5
1/25
1/125

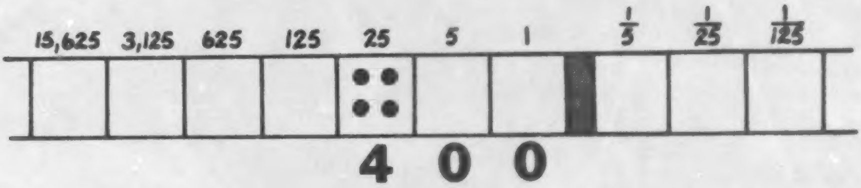
**T:** This is called a "base five" abacus. Who can put 100 on this abacus?

Accept any solution. Then lead to a solution which has at most four checkers on each board.


15,625
3,125
625
125
25
5
1
1/5
1/25
1/125

**T:** Who can write the base five name for 100?

Invite a student to write the number by aligning the digits of the base five number with the boards of the abacus.


15,625
3,125
625
125
25
5
1
1/5
1/25
1/125
4 0 0

NOTE: Read "**400**" in base five as "four, zero, zero, base five" instead of as "four hundred".

**T:** Why is this 100?

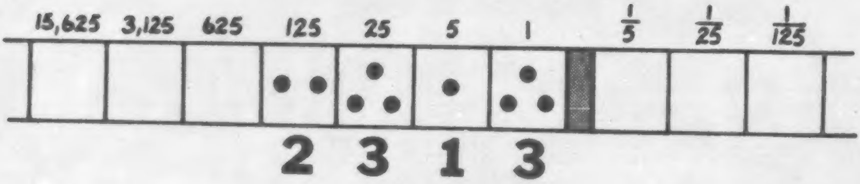
**T:** 4 x 25 = 100.

Draw this table on the board.

| Decimal writing | Base Five writing |
|---|---|
| 100 | 400 |
| 333 | |
| | |

# W 7

T: Who can put 333 on this abacus and can write its base five name?



T: In our regular, decimal writing of numbers we need ten digits, namely 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. How many digits do we need in base five?

S: Five: **0**, **1**, **2**, **3**, and **4**.

T: Why only five? What happens if I show a number by putting more than four checkers on a board of the abacus?

S: If there are five or more checkers on any board of the abacus, we can make a play by trading five checkers for one checker on the next board to the left.

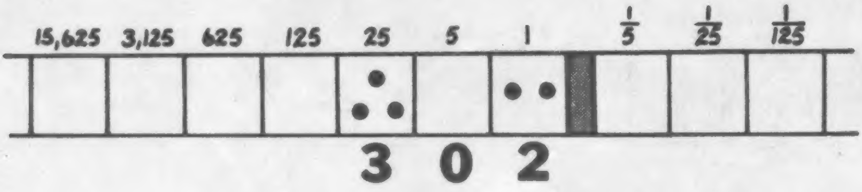
Add the following information to the table.

| Decimal writing | Base Five writing |
|---|---|
| 100 | 400 |
| 333 | 2313 |
| | 302 |
| | 21.3 |

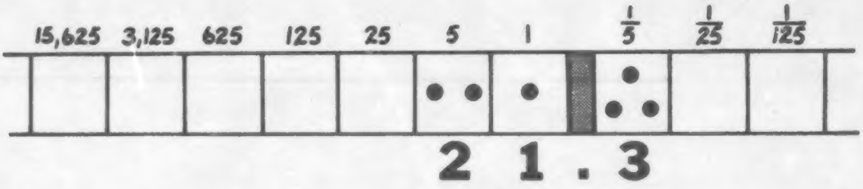
T (pointing to " **302** "): What decimal number is this?

S: 77, because (3 × 25) + 2 = 77.

Invite a student to put it on the abacus.


15,625
3,125
625
125
25
5
1
1/5
1/25
1/125
3 0 2

T (pointing to "**21.3**"): Who can put this number on the abacus?


15,625
3,125
625
125
25
5
1
1/5
1/25
1/125
2 1 . 3

T: What decimal number is this?

S: 11 $\frac{3}{5}$ or 11.6, since $\frac{1}{5}$ = 0.2 and $\frac{3}{5}$ = 0.6.

Add the information to the table.

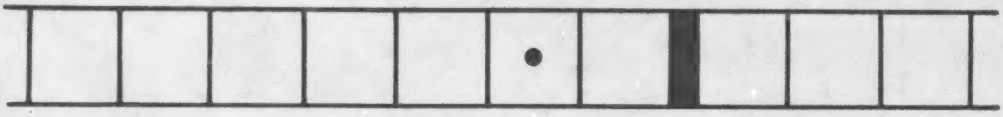
| Decimal writing | Base Five writing |
|---|---|
| 100 | 400 |
| 333 | 2313 |
| 77 | 302 |
| 11 $\frac{3}{5}$ or 11.6 | 21.3 |

Distribute copies of Worksheet N66 and ask students to complete it. After most of the students have finished the worksheet, invite some students to present their solutions.

# W 7

Exercise 2

Erase the labels from the abacus. Put one checker on the abacus as shown here.



T: I'll change the rule: one checker on a board is the same as ten checkers on the next board to the right. If this is the rule, what number is on the abacus?

S: 10.

T: This is now a base ten abacus. How many digits do I need to write numbers in base ten?

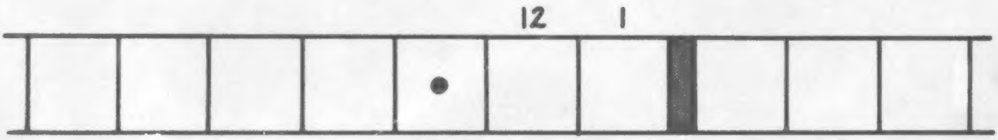
S: Ten: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

T: Will numbers written in base ten be familiar to you?

S: Yes, its' our usual way of writing numbers.

T: "Base ten" is another name for decimal.

Change the rule of the abacus again and indicate it in the following way.



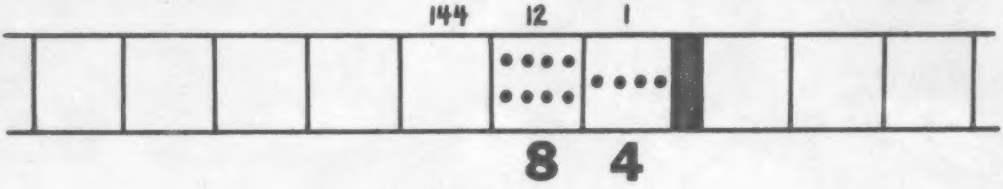
T: I've changed the rule: one checker on a board is the same as twelve checkers on the next board to the right. What number is on the abacus?

S: 144, because 12 × 12 = 144.

Label that board "144". Draw the following table on the board.

| Decimal writing | Base Twelve writing |
|---|---|
| 100 | |
| | |

T: Who can put 100 on this abacus?



NOTE: Read "**84**" in base twelve as "eight, four, base twelve".

T: Why is this 100?

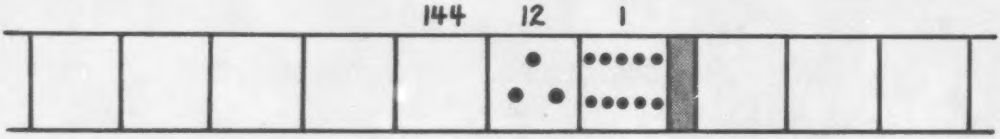
S: (8 × 12) + 4 = 100.

Add the following information to the table.

| Decimal writing | Base Twelve writing |
|---|---|
| 100 | 84 |
| 46 | |

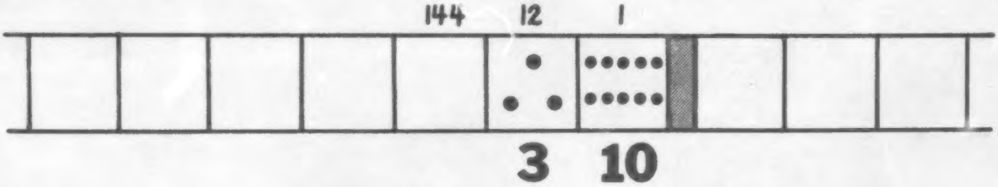
# W 7

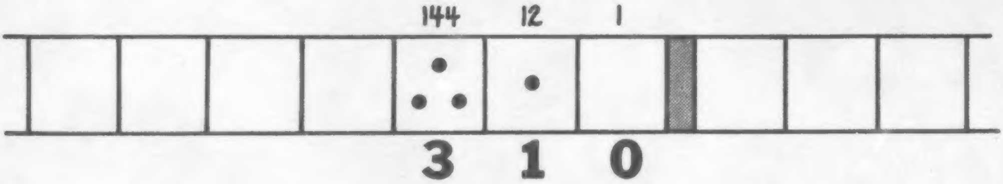
T: Who can put 46 on the abacus?



T: Who can write the base twelve name for 46?

NOTE: Students are very likely to suggest incorrectly "**310**" due to writing "**3**" beneath the twelves' board and "**10**" beneath the ones' board. If this occurs, ask the class how they would distinguish between the following two numbers written in base twelve. The decimal name for the first one is "46" and for the second one is "444"; they are different numbers.





T: What is the problem? Why is it difficult to write the base twelve name for 46?

Lead the class to observe that you cannot represent ten checkers on a board until you have a base twelve name for 10.

T: How many digits do we need to write base twelve numbers?

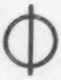
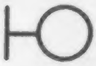
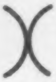
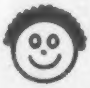
S: Twelve.

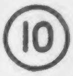
T: Yes, but we have only ten digits; 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. What numbers do we still need symbols for?

S: 10 and 11.

T: Let's invent a new symbol for ten for use in base twelve.

Invite several students to suggest new symbols for ten. Let the other students express their opinions of the suggested symbols. Students in one of our development classes suggested the following symbols:



After a few symbols have been drawn, present some of the desired characteristics of the new symbol; for example,

T: The new symbol should be simple to write and not look like other mathematical symbols, such as the numerals "0" through "9" or the operation symbols "+", "-", "x", or "÷". Also, the symbol need not remind us of the numeral "10".

Write "6" on the board.

T (pointing to "6"): Nor does the symbol have to suggest quantity; for example, does the way this symbol is written suggest six objects?

S: No, but we have learned to think "six" when we see that symbol.

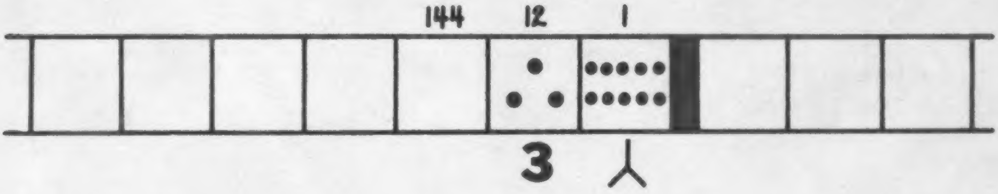
T: Our new symbol for ten need not remind us of ten objects or of the numeral "10". It can be entirely new.

# W 7

Using these criteria, let the students discuss the symbols they have already suggested and invite them to suggest a few more symbols. Then ask the class to select the symbol they prefer. For example, one of our development classes chose " ⅄ ".

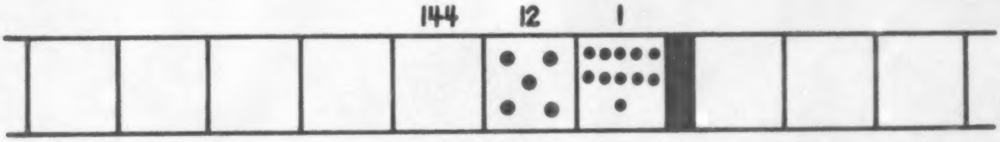
T: Now who can write the base twelve name for 46?

The student who volunteers should use the new symbol chosen by your class; for example,



| Decimal writing | Base Twelve writing |
|---|---|
| 100 | 84 |
| 46 | 3⅄ |

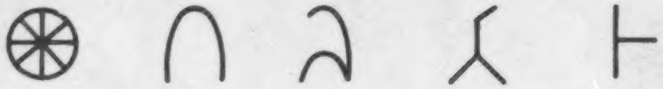
T: Who can put 71 on this abacus?



T: Can we write the base twelve name for 71?

S: Not yet, we need a new symbol for eleven.

Let students suggest symbols for eleven. The students in one of our development classes drew the following symbols:



Let the class select one of their symbols, e.g. "∩", as the base twelve name for eleven.

T: Who can write the base twelve name for 71?

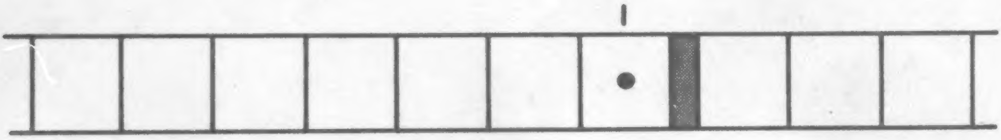
| Decimal writing | Base Twelve writing |
|---|---|
| 20 | 18 |
| 190 | 13⅄ |
| 71 | 5∩ |

Exercise 3 (optional)

This discussion of positional notation is optional. You may wish to include it in this lesson or at some future time.

T: Today we looked at a base five abacus and at a base twelve abacus. In other lessons you have used a base two abacus and a base three abacus. What is similar and what is different about these abacuses?

One point you should discuss is that one checker on the first board to the left of the wide bar is always 1.



# W 7

The rule for the abacus is arbitrary. However, once the rule is chosen (for example, five checkers on a board is the same as one checker on the next board to the left), the values of the boards are determined by the multiplication rule (e.g., 5, 25, 125, 625, $\frac{1}{5}$, $\frac{1}{25}$, and so on).

Also, the students should observe that the rule we choose for the abacus determines the number of digits we need to write names for numbers on that abacus. For example, we need five numerals ("0", "1", "2", "3", "4") to write base five names.

T: Why do you think we usually write base ten names for numbers instead of some other base?

It should be brought out in the discussion that man has ten fingers and that this led to a base ten method of writing numbers.

T: What bases do you think other earlier cultures may have used? Why?

After the class has suggested a few possibilities, you might mention that Romans used a mixture of base five and base ten, that the Mayans of Central America used a mixture of base five and base twenty, and that the Babylonians used a mixture of base ten and base sixty, from which our method of keeping time originated (60 seconds = 1 minute; 60 minutes = 1 hour). Also, today some computers use base sixteen and we will be discussing this system in a later lesson (the third lesson on ELECTION IN THE NUMBER WORLD.)

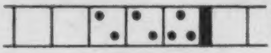
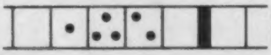
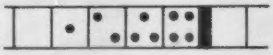
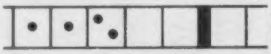
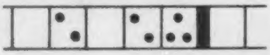
Write this expression on the board.

positional notation

T: Each of these ways of writing numbers is an example of positional notation. Why do you think we describe the notations as "positional"?

S: The position of a checker on an abacus determines the value of that checker.

Name ____________________ N 66

| Decimal Writing | | Base Five Writing |
|---|---|---|
| 63 |  | 223 |
| 210 |  | 1320 |
| 194 |  | 1234 |
| 800 |  | 11200 |
| 263 |  | 2023 |

Lesson 2: The Storybook ELECTION IN THE NUMBER WORLD

NOTE: Because there are several bases referred to in this lesson, it is necessary to be very careful in how you read numerals. For this reason, we often indicate how to read a numeral.

Do not distribute copies of the storybook until after Exercise 1.

Exercise 1

T: I'm going to put a number on the board, but I'm not going to tell you what kind of writing I'm using.

Write "**213**" on the board.

**213**

T: What number is this?

S: Two hundred thirteen.

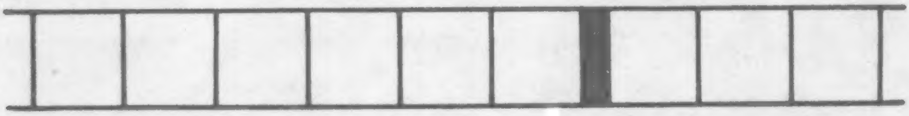
T: It could be two hundred thirteen. Could it be any other number?

Let the class discuss this. If no one suggests that you could be writing in a base other than base ten, proceed as follows.

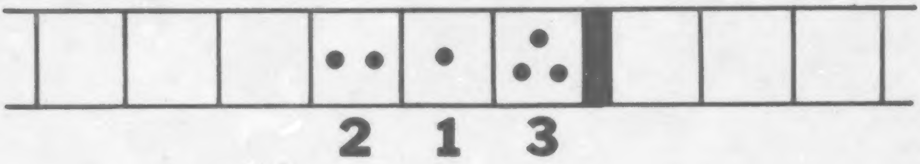
T: I'll give you a hint. Do you remember putting numbers on different kinds of abacuses?

Your students should recall that they have used abacuses for several bases, for example, base two, base three, base five, and base twelve. Then the class should realize that "213" could name several numbers depending upon which abacus or base writing is being used.

Draw an abacus on the board.



T: Who can use checkers to put "**213**" (read: two, one, three) on the abacus regardless of which kind of abacus it is?

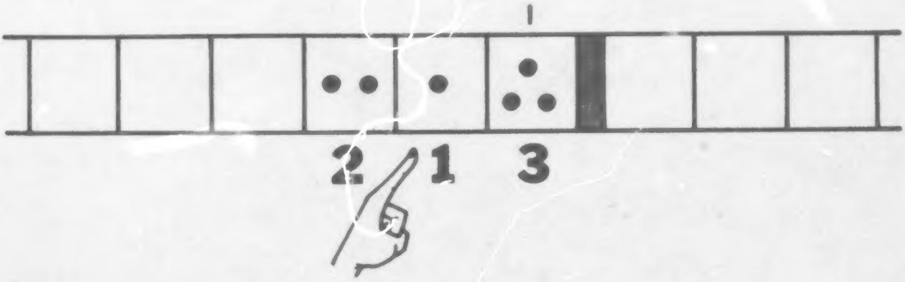


Point to the board with three checkers.

T: What is the value of this board?

S: One.

Label the board "1". Point to the board with one checker.



T: Could the value of this board be five?

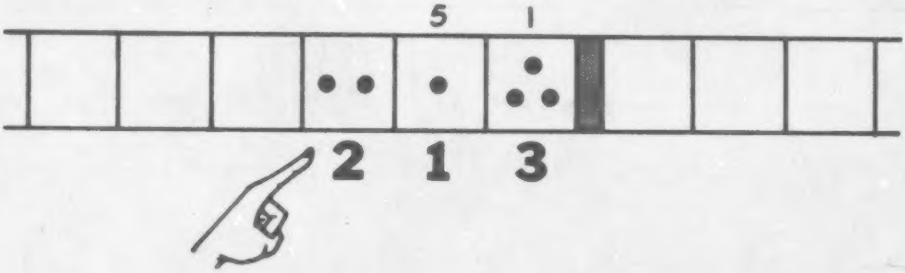
S: Yes.

T: If so, what would be the rule of the abacus?

S: You can trade five checkers on a board for one checker on the next board to the left.

# W 10

Label that board "5". Point to the board with two checkers.



T: What is the value of this board?

S: Twenty-five, since 5 x 5 = 25.

Label that board "25".

T: Therefore, what number is on the abacus?

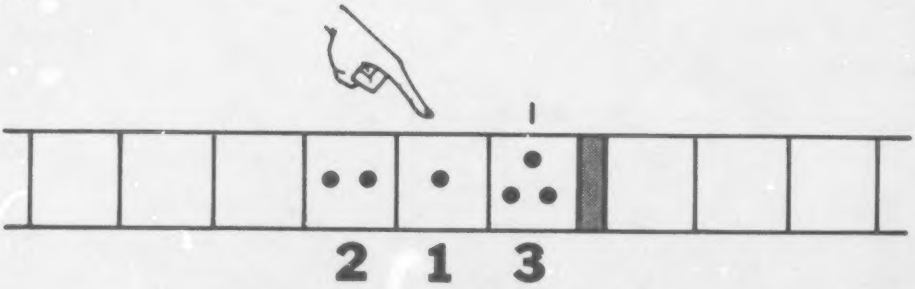
S: 58, since (2 x 25) + (1 x 5) + (3 x 1) = 58.

Write the following information on the board and read it to the class.

Base Five writing Decimal writing

$$213_{five} = 58_{ten}$$

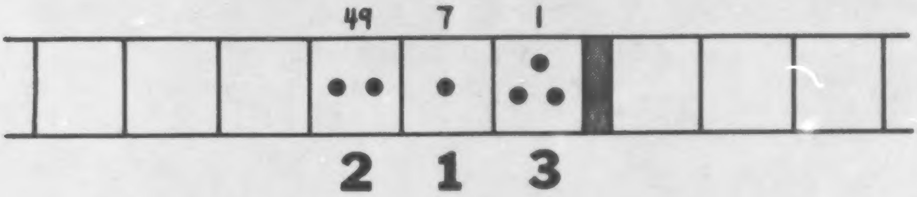
Erase the labels "5" and "25" from the abacus and point to the board with one checker.



T: Could the value of this board be seven?

S: Yes.

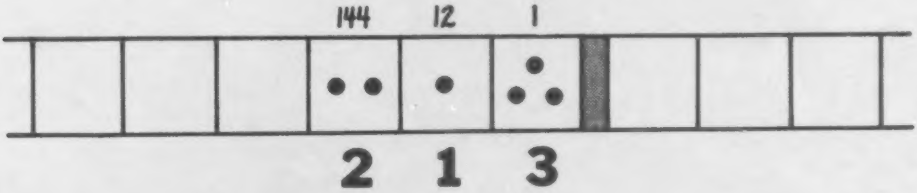
In a similar manner as above, ask students to determine the equivalent number in decimal writing to **213** (read: two, one, three) in base seven.



Base Seven writing Decimal writing

$$213_{seven} = 108_{ten}$$

Repeat the activity with base twelve and obtain the following result.



Base Twelve writing Decimal writing

$$213_{twelve} = 303_{ten}$$

T: We've looked at **213** (read: two, one, three) as a number written in base five, in base seven, and in base twelve. Could "**213**" be the name of a number written in other bases?

It could be the writing of a number in any base larger than three. Accept several student suggestions then ask:

T: Could "**213**" be the base two name of a number?

S: No, only "0" and "1" are used to write base two names for numbers.

Similarly conclude that "**213**" cannot be the base three name of a number since only "0", "1", and "2" are used in base three writing.

Write this expression on the board.

positional notation

T: Writing numbers in base five, in base seven, and in base twelve are all examples of positional notation. We call it "positional" because the position of a digit in a number or of a checker on an abacus determines its value.

Exercise 2

Distribute copies of the Storybook ELECTION IN THE NUMBER WORLD.

Pages 1-2

Read pages 1-2 and write the following information on the board.

| Fractional Writing | Positional Writing |
|---|---|
| $\frac{7}{5} + \frac{3}{8}$ | $3.2 + 17.6$ |

T: On page 2, Nabu is considering a positional way of writing numbers and the fractional way. I have written two calculations on the board, one with fractions and one with decimal writing. Which calculation is easier to do?

Even without doing the calculations, the students should agree that adding with decimal writing of numbers would be easier than adding with fractions.

T: One advantage of a positional notation is that it is usually easier to add numbers with it.

Pages 3-7

Read pages 3-7 together.

T: Look at the blue arrow road on page 7. What could the blue arrows be for?

S: 10 ×, because 10 × 0.1 = 1; 10 × 1 = 10; 10 × 10 = 100; and so on.

T: Find 1,000 on the arrow road. Let's start at 1,000 and read all the larger numbers in order.

Giving help when needed, invite students to read the numbers in order:

ten thousand, one hundred thousand,
one million (1,000,000), ten million, one hundred million,
one billion (1,000,000,000), ten billion, one hundred billion,
one trillion (1,000,000,000,000), ten trillion, one hundred trillion,
one quadrillion (1,000,000,000,000,000), and ten quadrillion.

Pages 8-9

Read page 8 together. Then write these expressions on the board.

Decimal ________
Base Two or Binary ________
Fractions ________

T: The numbers are going to vote on which method of writing numbers they prefer. Let's take a vote in this class. If we were planning to use only one of these ways of writing numbers, which would you vote for? Why?

# W 10

After a class discussion, take a class vote.

T: The numbers voted and their results are on page 9.

Ask a student to read page 9.

T: Which system received the most votes?

S: Decimal writing.

T: Who can explain these results: 95%; 4.7%; and 0.3%?

Some students are likely to have encountered percents elsewhere. Let these students discuss the voting results. For example, they may realize that in this situation 95% means 95 out of every 100 were for decimal writing. Keep the discussion very brief since there will be specific lessons on percent in Part VI of the CSMP Intermediate Grades program.

Pages 10-21

As you and the class read pages 10-21, stop briefly to solve each problem by asking students to point to the answer in their storybooks.

Pages 22-31

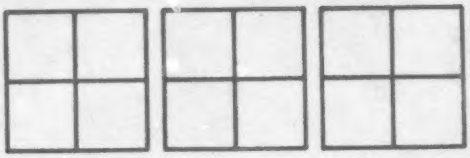
Read pages 22-31 together. Briefly discuss the ideas as you wish, but try to finish the storybook in this one lesson.

If possible, we suggest that students be allowed to keep their copies of the storybook. They will not be needed in future lessons.

## Lesson 3: The Two-Sixteen Code

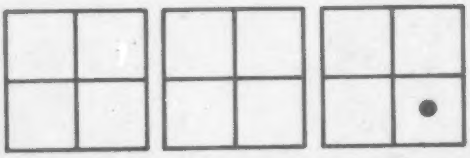
### Exercise 1

Display three Minicomputer boards.



T: Today, I'll show you a new code for putting numbers on the Minicomputer.

Put a checker on the 1-square.



T: What number is this?

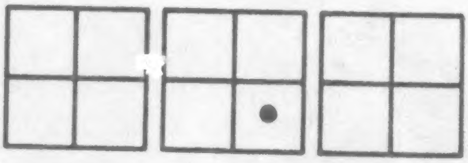
S: 1.

Move the checker from square to square on the ones' board to show 2, 4, and 8. Emphasize that the number doubles as the checker is moved from square to square.

T: Do you remember the binary abacus? The four squares on the ones' board of the Minicomputer are like the first four squares of a binary abacus since their values are 1, 2, 4, and 8.

# W 13

Put one checker on the Minicomputer as shown here.



T: What number is this?

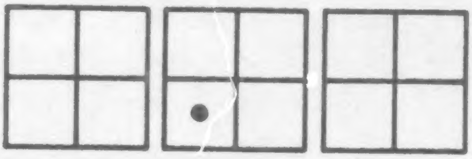
S: 10.

T: Yes, with the usual Minicomputer code this is 10. But today we'll use a new code based on doubling. If we continue the doubling pattern, what number would this be?

S: 16, since 2 x 8 = 16.

If no one suggests 16, do so yourself.

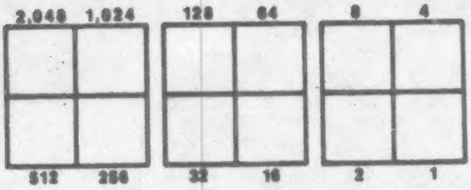
Move the checker to the red square of the middle board.



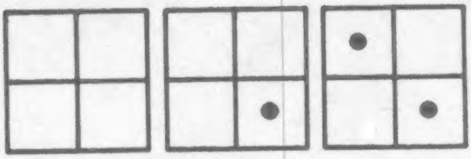
T: Continuing our pattern, what number is this?

S: 32.

Continue moving the checker from square to square, asking students to identify each number. The value of each square is shown in the next picture. Do not label the squares on the demonstration Minicomputer.


2,048
1,024
128
64
8
4
512
256
32
16
2
1

**T:** Using our new code, who can put 25 on the Minicomputer?

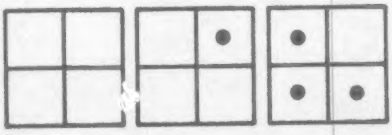


= 25

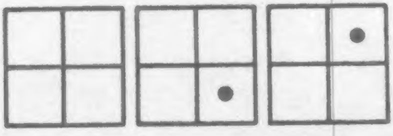
**T:** Why is this 25?

**S:** 16 + 8 + 1 = 25.

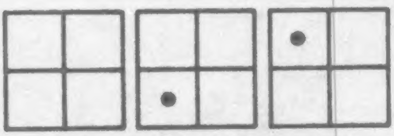
Invite students to put the following numbers on the Minicomputer using the new code.



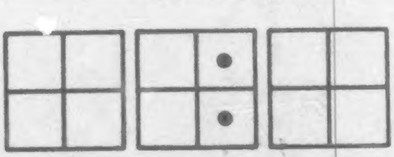
= 64 + 8 + 2 + 1 = 75



= 16 + 4 = 20



= 32 + 8 = 40

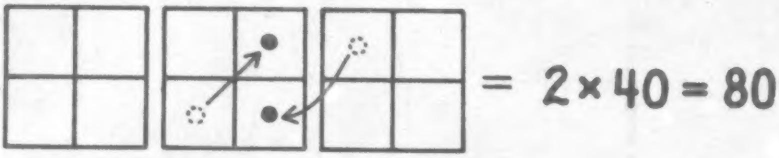


= 64 + 16 = 80

# W 13

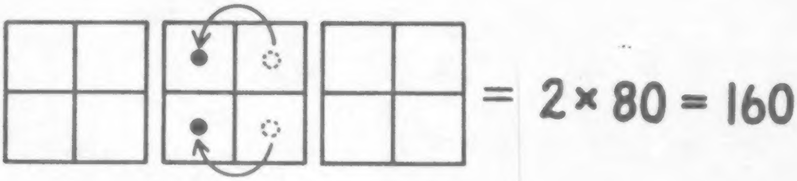
T: Does anyone see a pattern in the way 20, 40, and 80 were put on the Minicomputer?

A student is likely to notice that to double a number on this Minicomputer, each checker is simply moved to the square of the next higher value; for example, to double 40 the checkers can be moved as illustrated below.



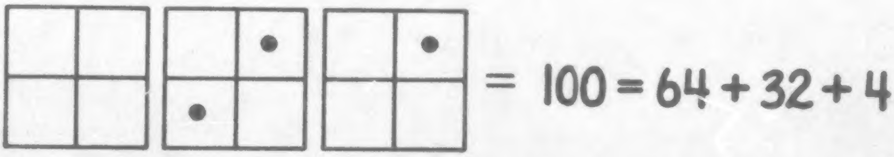
T: Can someone use this pattern to put 160 on the Minicomputer?

The student who volunteers should move each checker to the square of the next higher value.

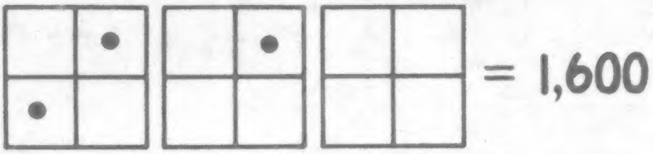


Review this doubling method by putting 20 on this Minicomputer and moving the checkers quickly to put on successively 40, 80, and 160.

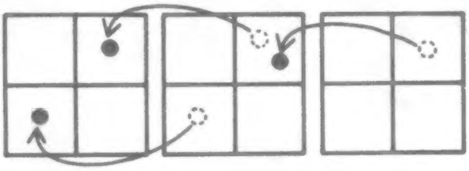
T: Who can put 100 on the Minicomputer?



T: Put 1,600 on the Minicomputer.



Ask the students to compare the configuration for 100 and 1,600. They should notice that each checker has been moved to the square of the same color on the next board to the left.

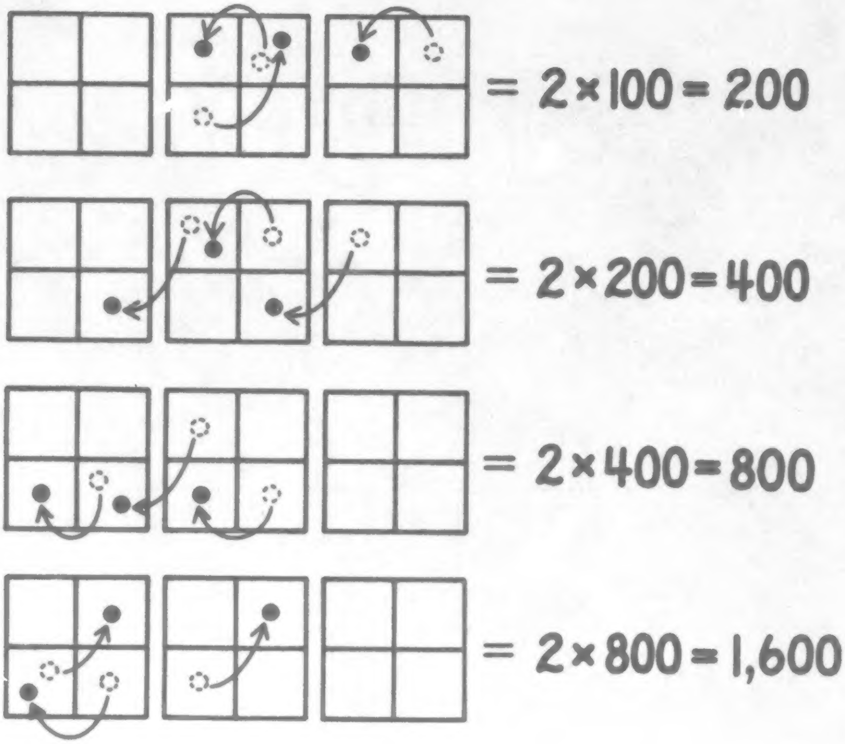


T: Who can explain this pattern?

S: Moving the checkers one board to the left multiplies the number by 16; 16 x 100 = 1,600.

S: Moving the checkers one board to the left is the same as doubling the number four times; 2 x followed by 2 x followed by 2 x followed by 2 x is the same as 16 x.

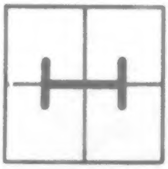
You may wish to demonstrate the above explanation by doubling 100 four times on the Minicomputer using the method of moving each checker to the next square of higher value. See the next illustration.


= 2×100 = 200
= 2×200 = 400
= 2×400 = 800
= 2×800 = 1,600

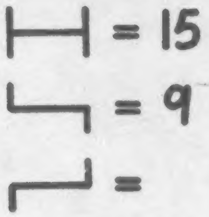
Exercise 2

T: When Nabu saw this new Minicomputer code, he invented a new way of writing numbers.

Without comment, draw the following picture at one end of the board.



Write this information on the board.

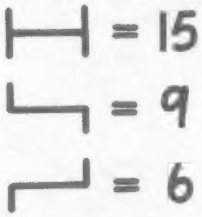


NOTE: See the end of this exercise for an explanation of this code.

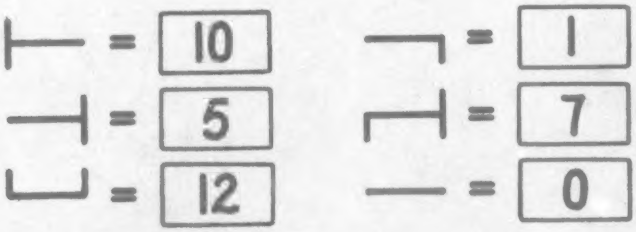
T: I will not explain Nabu's new code to you. See if you can crack his code.

What number is this (point to "┌─┘")?

Walk around the room and let students whisper their guesses in your ear. If correct, simply say, "Yes, that's correct." If incorrect, say, for example, "No, 5 is not the number." After listening to several students, record that ┌─┘ = 6.

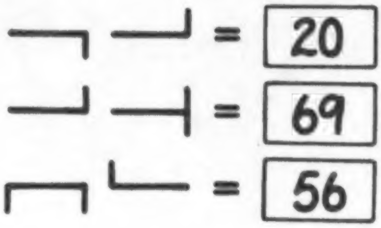


Continue the activity with the problems below (answers are shown in the boxes). Tell students who need a hint that Nabu's code has something to do with the new Minicomputer code. Ask students who think they understand the code to keep the secret to themselves for a while.



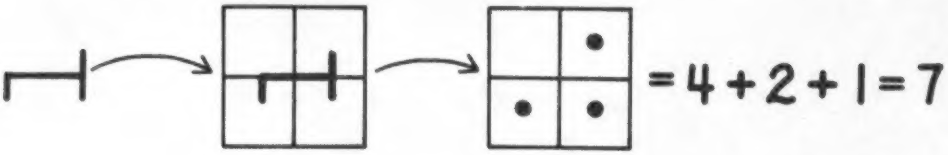
# W 13

For the following examples, remind the students to use the new Minicomputer code that they just learned.

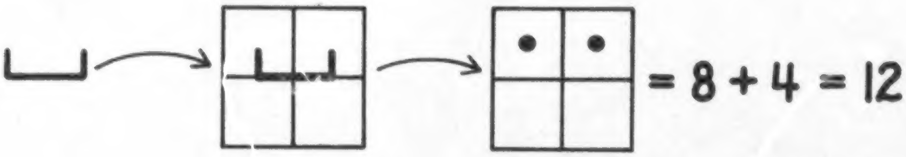


T: Who can explain Nabu's new code? (Point to "┌─┤".) Why is this 7?

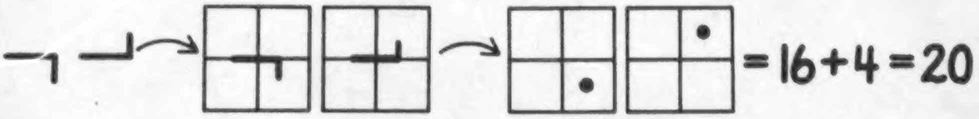
Let students explain how each symbol of the new code indicates where to put checkers on a Minicomputer board. The following sequence of pictures shows how a student might explain why ┌─┤ = 7.



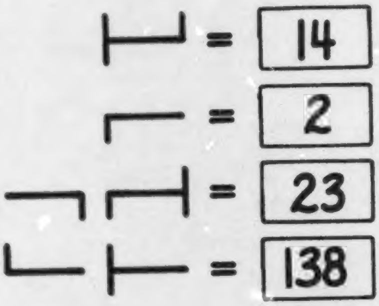
T (pointing to "└─┘"): Why is this 12?



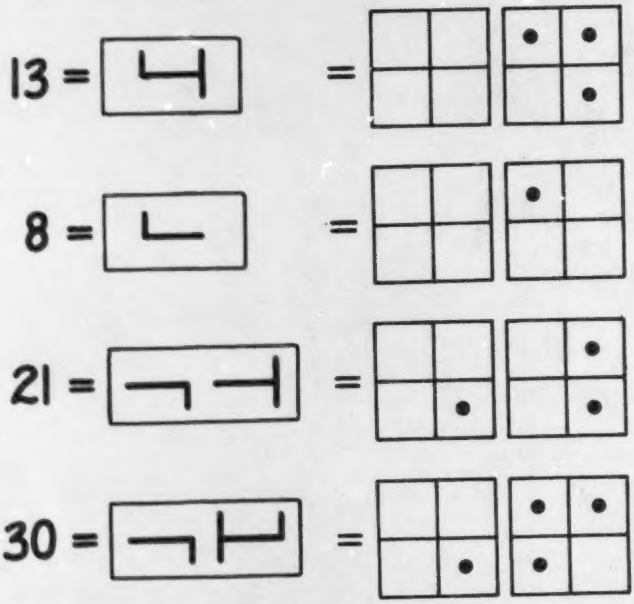
T (pointing to "─┐ ─┘"): Why is this 20?



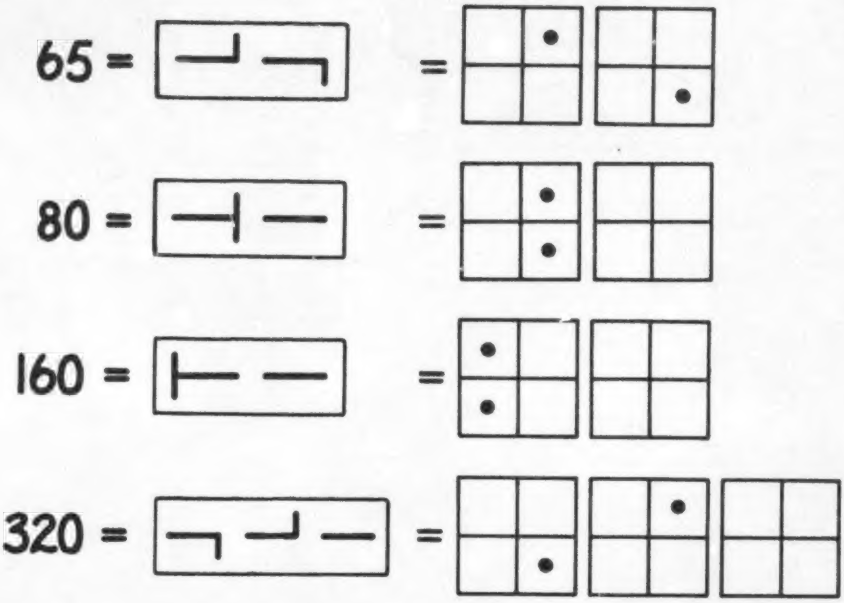
After the code has been explained, ask the class to decode a few more numbers; for example,



Invite students to write some numbers using Nabu's new code. If necessary, ask them to put the number on the Minicomputer before writing the code. Answers are shown in the boxes.



# W 13



Point to some of the symbols used in Nabu's code.

T: We have used many different symbols to write numbers in Nabu's code. On your paper, write all the different symbols that are used in Nabu's code.

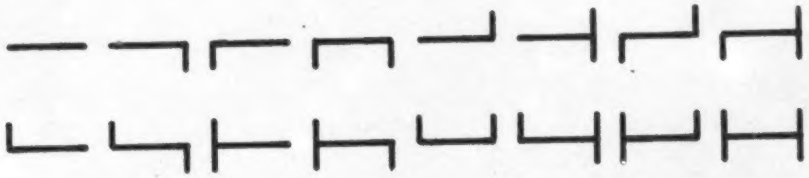
Let the students work independently for a while. Encourage them to write the symbols in some systematic order.

T: How many different symbols are there in Nabu's code?

S: Sixteen.

Invite a student who has found all sixteen symbols to write them in order from the smallest number (0) to the largest number (15).

If no student has found all sixteen, invite a student who has found the most symbols to write them on the board. Then challenge the other students to find the missing symbols.



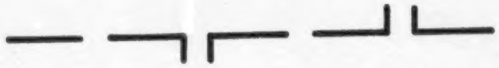
T: With these sixteen symbols we can write any number. So this is a way of writing numbers in base sixteen.

Which are the simplest symbols in Nabu's code?

S: The symbol for 0, "—"; it uses only one line segment.

S: The symbols for 1, 2, 4, and 8; they use only two line segments each.

Write these five symbols on the board.



T: How are the other symbols made from these simple symbols? For example, how could you make the symbol for 6?

S: Combine the symbols for 2 and 4.

Write the symbol for 6, "┌─┘", on the board. Show how this symbol could be formed by combining the symbol for 2, "┌─", and the symbol for 4, "─┘".

T: Can the other symbols be formed in similar ways?

# W 13

Let the students suggest a few more examples such as: 11 is formed from └— (8), ┌— (2), and —┐ (1).

T: So, the most important symbols are those for 1, 2, 4, and 8. Do you remember an abacus on which 1, 2, 4, and 8 are important numbers?

S: Yes, the binary or base two abacus.

T: So Nabu's code uses ideas from both base sixteen and base two.

# ARCADE
# OF
# PROBLEMS
# #4

# ARCADE OF PROBLEMS #4

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.* You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems.

A brief collective lesson is suggested to begin the second workbook lesson. This should be a short warm-up activity to introduce two by two tables for the operations $x_5$ and $T_0$.

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* See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

# W8&9

## Lesson 1

Give each student a copy of "Arcade of Problems #4" and let them work on their own.

If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

## Lesson 2

Draw the following table on the board.

| $\times_5$ | 2 | 9 |
|---|---|---|
| 8 | b | |
| 5 | | |

Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: This table is a small part of the operation table for $\times_5$ (read: multiplication modulo five).
What number goes here (point to b)?

S: 1, because $8 \times_5 2 = 1$.

T: How did you do the calculation?

S: $8 \times 2 = 16$ and when you divide 16 by 5, the remainder is 1.

S: $8 \times 2 = 16$ and I subtracted fives from 16 until I got to 1.

T: Both methods are correct. If you subtract fives, how do you know how many fives to subtract?

S: You stop subtracting fives when you get one of the numbers 0, 1, 2, 3, or 4.

Write "1" in the square for "8 $\times_5$ 2". Invite students to complete the table.

| $\times_5$ | 2 | 9 |
|---|---|---|
| 8 | 1 | 2 |
| 5 | 0 | 0 |

Draw the following table on the board.

| $T_D$ | 15 | 28 |
|---|---|---|
| 3 | b | |
| 7 | c | |

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to "$T_D$"): This table is the truth table for the relation D, "is a divisor of." What number goes here (point to b)?

S: 1, because 3 is a divisor of 15.

If necessary, remind the class that we write "1" for a true statement and "0" for a false statement.

Write "1" in the square for "3 $T_D$ 15".

T (pointing to c): What number goes here?

S: 0, because 7 is not a divisor of 15.

# W 9

Invite the students to complete the table.

| $T_D$ | 15 | 28 |
|---|---|---|
| 3 | 1 | 0 |
| 7 | 0 | 1 |

Return the students' copies of "Arcade of Problems #4". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

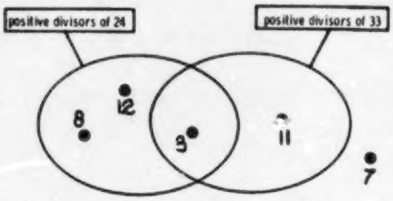
If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

## Arcade of Problems # 4

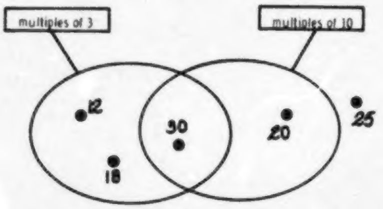
Put each of these numbers in the string picture.

3 7 8 11 12


positive divisors of 24
positive divisors of 33
8
12
3
11
7

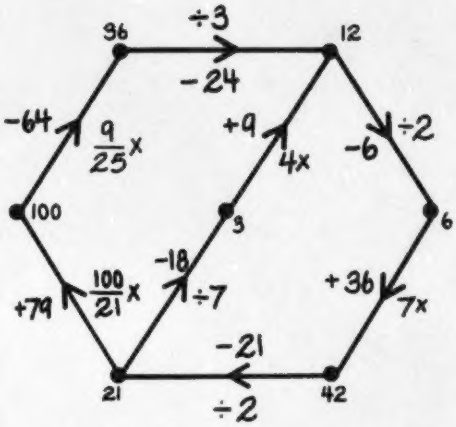
Put each of these numbers in the string picture.

12 18 20 25 30


multiples of 3
multiples of 10
12
18
30
20
25

2

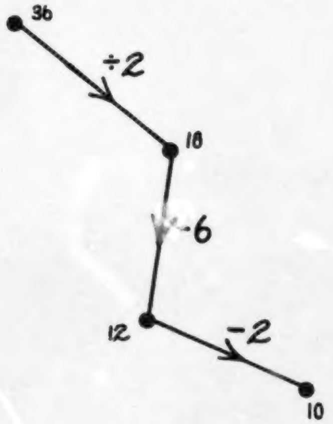
Label the arrows; try to label some arrows in two ways.
Many solutions are possible.


36
÷3
-24
12
-64
9/25 x
+9
4x
-6
÷2
100
3
6
-18
100/21 x
÷7
+36
7x
+79
-21
21
42
÷2

3

(Other solutions are possible.)

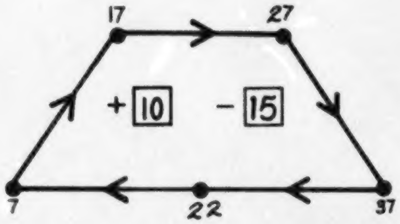
Build an arrow road from 36 to 10. Each arrow must be for one of these:
+2, +6, -2, -6, x2, x6, ÷2, ÷6

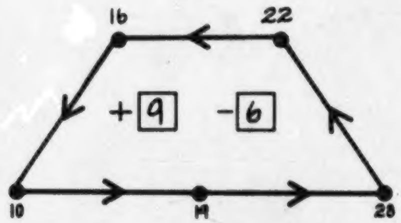

36
÷2
18
-6
12
-2
10

4

(Many solutions are possible. The shortest roads have three arrows.)

Fill in the boxes for the arrows and label the dots.

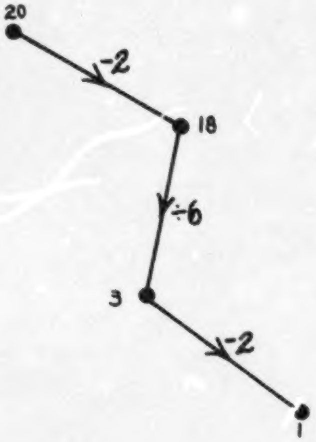

17
27
+ 10
- 15
7
22
37


16
22
+ 9
- 6
10
19
28

5

Build an arrow road from 20 to 1. Each arrow must be for one of these:

+2, +6, −2, −6, x2, x6, ÷2, ÷6



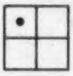
(Many solutions are possible. The shortest roads have three arrows.)

---

Poy is a secret number.

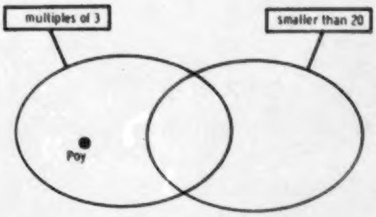
**First clue**

Poy can be put on this ones' board of the Minicomputer by adding exactly one ⑦-checker.



Poy could be 15, 22, 36, or 64.

**Second clue**

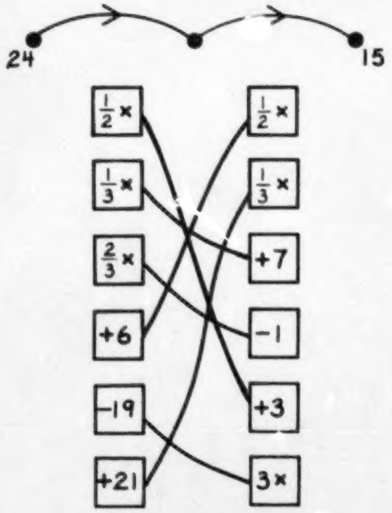


multiples of 3
smaller than 20
Poy

Poy is 36.

---

Pair the tags.



24
15
$\frac{1}{2}$×
$\frac{1}{3}$×
$\frac{2}{3}$×
+6
−19
+21
$\frac{1}{2}$×
$\frac{1}{3}$×
+7
−1
+3
3×

---

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 in each box to make the calculations correct. You may use a digit more than once.

Add

$$\begin{array}{r} 7\,6\,\boxed{6} \\ +\ \boxed{3}\,\boxed{4}\,8\,3 \\ \hline 4\,2\,\boxed{4}\,9 \end{array}$$

Add

$$\begin{array}{r} \boxed{7}\,0\,9\,3 \\ \boxed{5}\,7\,4 \\ +\ 7\,2\,5\,\boxed{5} \\ \hline 1\,4\,9\,\boxed{2}\,2 \end{array}$$

Add

3,786 + 19 + 807

4,612

Subtract

$$\begin{array}{r} \boxed{7}\,8\,0 \\ -\ 2\,\boxed{4}\,6 \\ \hline 5\,3\,\boxed{4} \end{array}$$

Subtract

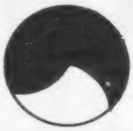
63,704 − 838

62,866

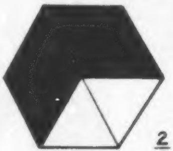
Subtract

$$\begin{array}{r} \boxed{5}\,5\,6\,\boxed{1} \\ -\ 1\,\boxed{6}\,\boxed{5}\,7 \\ \hline 3\,9\,0\,4 \end{array}$$

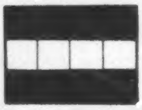
Color two-thirds of each shape red. Write another name for $\frac{2}{3}$ as suggested by the picture.



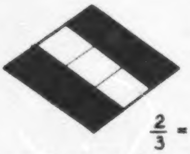
$\frac{2}{3}$



$\frac{2}{3} = \frac{4}{6}$



$\frac{2}{3} = \frac{8}{12}$



$\frac{2}{3} = \frac{6}{9}$

Fill in each box with a whole number.

$$\frac{2}{3} = \frac{\boxed{12}}{18} = \frac{10}{\boxed{15}} = \frac{\boxed{40}}{60} = \frac{20}{\boxed{30}}$$

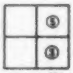
(Other colorings are possible.)

Put each type of number on the ones' board of the Minicomputer by adding exactly one of these checkers.

● ③

Many answers are possible.

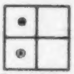
An odd number



= 23

(21 is another solution.)

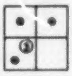
A positive divisor of 30



= 6

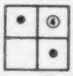
(1, 2, and 10 are other solutions.)

A multiple of 10



= 20

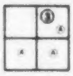
A multiple of 5



= 25

(20 is another solution.)

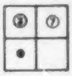
A positive prime number



= 5

(17 another solution.)

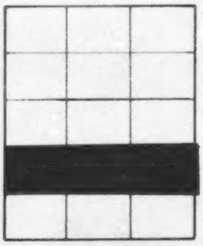
A number larger than 40



= 54

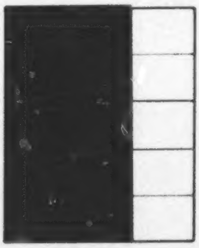
(42 is another solution.)

Color one-fifth of this region red and fill in the box.



$\frac{1}{5} = \frac{\boxed{3}}{15}$

Color two-thirds of this region red and fill in the box.



$\frac{2}{3} = \frac{\boxed{10}}{15}$

Complete.

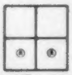
$$\frac{1}{5} + \frac{2}{3} = \frac{\boxed{3}}{15} + \frac{\boxed{10}}{15} = \underline{\frac{13}{15}}$$

(Other colorings are possible.)

Gat is a secret number.

First clue

Gat can be put on this Minicomputer by adding exactly one of these checkers.

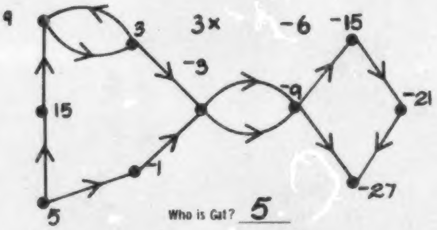


^ ④

Gat could be $^{-}4$, $^{-}5$, $^{-}7$, $^{-}11$, 1, 5, 13, or 29.

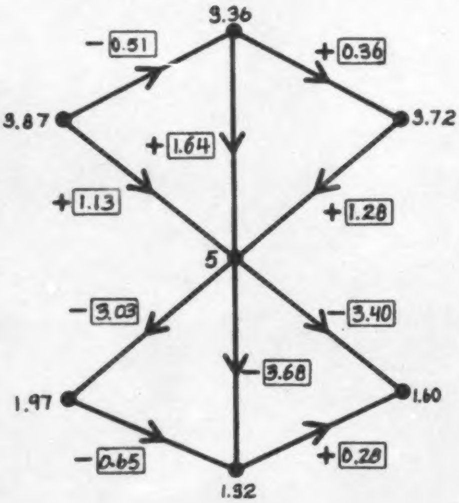
Second clue

Gat is in this arrow picture.



Who is Gat? 5

Fill in the box for each arrow.


3.36
$-\boxed{0.51}$
$+\boxed{0.36}$
3.87
3.72
$+\boxed{1.64}$
$+\boxed{1.13}$
$+\boxed{1.28}$
5
$-\boxed{3.03}$
$-\boxed{3.40}$
$\boxed{3.68}$
1.97
1.60
$-\boxed{0.65}$
$+\boxed{0.28}$
1.32

Fill in each box with a whole number.

$$\frac{1}{2} = \frac{\boxed{3}}{6} = \frac{\boxed{7}}{14} = \frac{6}{\boxed{12}} = \frac{20}{\boxed{40}}$$

$$\frac{1}{3} = \frac{\boxed{3}}{9} = \frac{\boxed{10}}{30} = \frac{4}{\boxed{12}} = \frac{6}{\boxed{18}}$$

$$\frac{1}{4} = \frac{\boxed{5}}{20} = \frac{\boxed{25}}{100} = \frac{3}{\boxed{12}} = \frac{11}{\boxed{44}}$$

$$\frac{3}{4} = \frac{\boxed{15}}{20} = \frac{\boxed{21}}{28} = \frac{9}{\boxed{12}} = \frac{30}{\boxed{40}}$$

Complete the tables.

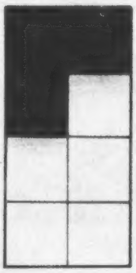
| $+_{10}$ | 6 | 2 |
|---|---|---|
| 7 | 3 | 9 |
| 19 | 5 | 1 |

| $\times_5$ | 4 | 7 |
|---|---|---|
| 2 | 3 | 4 |
| 6 | 4 | 2 |

| $T_D$ | 18 | 30 |
|---|---|---|
| 6 | 1 | 1 |
| 10 | 0 | 1 |

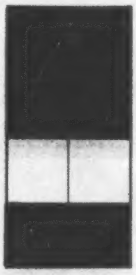
| $T_{\equiv_2}$ | 9 | 12 |
|---|---|---|
| 5 | 1 | 0 |
| 20 | 0 | 1 |

Color three-eighths of this region red.



$\frac{3}{8}$

Color three-fourths of this region red and fill in the box.



$\frac{3}{4} = \frac{\boxed{6}}{8}$

Complete.

$$\frac{3}{8} + \frac{3}{4} = \frac{9}{8} = 1\frac{1}{8}$$

(Other colorings are possible.)

Daf and Laf are secret numbers.

First clue

Daf and Laf are two of these numbers.

$\sqcap$: Greatest common divisor

$\sqcup$: Least common multiple

$20 \sqcap 12 = $ 4

$18 \sqcap 45 = $ 9

$27 \sqcap 20 = $ 1

$8 \sqcup 6 = $ 24

$4 \sqcup 7 = $ 28

$10 \sqcup 4 = $ 20

Second clue

$Daf =_3 2$ $Laf =_3 0$

Daf is 20.

Laf could be 9 or 24.

Third clue

$Daf + Laf =_4 1$

Laf is 9.

---

Fill in each box with a whole number.

Example: $4^3 = 4 \times 4 \times 4 = 64$

$7^2 = \boxed{49}$

$5^2 = \boxed{25}$

$7^3 = \boxed{343}$

$5^3 = \boxed{125}$

$20^3 = \boxed{8{,}000}$

$5^4 = \boxed{625}$

$4^{\boxed{[illegible]}} = 16$

$\boxed{11}^2 = 121$

$3^{\boxed{[illegible]}} = 81$

$\boxed{30}^2 = 900$

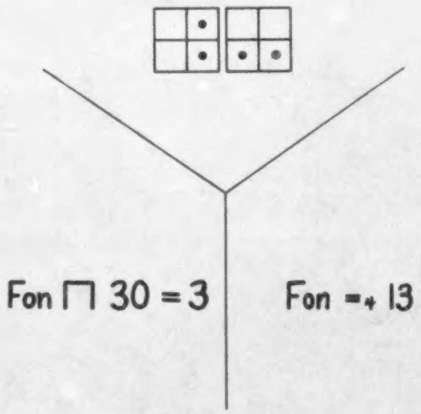
$2^{\boxed{[illegible]}} = 512$

$\boxed{7}^3 = 343$

---

Fon is a secret number.

Fon can be shown on this Minicomputer by taking off exactly two checkers.



Who is Fon? 9

---

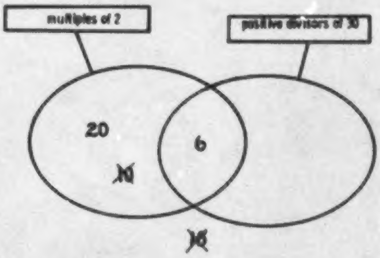
The red tag is one of these:

- multiples of 2
- multiples of 3
- multiples of 5
- smaller than 12
- positive divisors of 20
- positive divisors of 30

The blue tag is one of these:

- multiples of 2
- multiples of 3
- multiples of 5
- smaller than 12
- positive divisors of 20
- positive divisors of 30

Label the strings.



Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Multiply

$$\begin{array}{r} 938 \\ \times\ 706 \\ \hline 662{,}228 \end{array}$$

Multiply

$$\begin{array}{r} \boxed{6}\,5\,\boxed{3} \\ \times\ \ \boxed{4} \\ \hline 5\,\boxed{8}\,7\,7 \end{array}$$

Divide

$$\boxed{2}\boxed{5}\ \text{R}\ \boxed{1}\boxed{6}$$
$$\begin{array}{r|r} 23\,)\,5\,9\,\boxed{1} & \\ -\boxed{4}\boxed{6}\boxed{0} & \boxed{2}\,0 \\ \hline \boxed{1}\boxed{3}\,1 & \\ -\boxed{1}\boxed{1}\boxed{5} & \boxed{5} \\ \hline \boxed{1}\boxed{6} & \end{array}$$

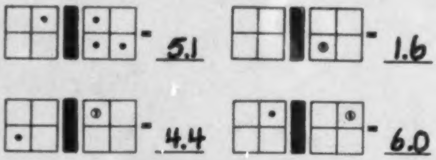
Divide

$$\boxed{4}.\boxed{2}\boxed{5}\ \text{R}\ 0.\boxed{0}\boxed{7}$$
$$15\,)\,63.82$$

Fia is a secret number.

**First clue**

Fia is one of these numbers.



= 5.1

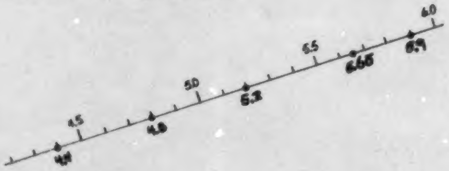
= 1.6

= 4.4

= 6.0

**Second clue**

Fia is one of these dots. Label the dots.


4.5
5.0
5.5
6.0
4.4
4.8
5.2
5.65
5.9

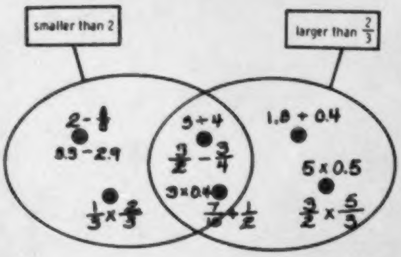
Who is Fia? 4.4

Label the dots. Some of the numbers have two names listed. Write both names for those numbers beside their dots.

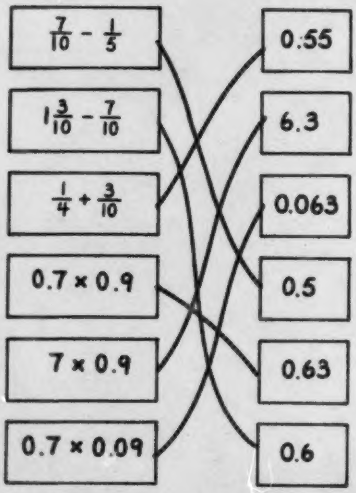
$3 \div 4$ $1.8 + 0.4$ $2 - \frac{8}{5}$

$\frac{1}{3} \times \frac{2}{3}$ $\frac{3}{2} \times \frac{5}{3}$ $\frac{3}{2} - \frac{3}{4}$ $\frac{7}{10} + \frac{1}{2}$

$3.3 - 2.9$ $5 \times 0.5$ $3 \times 0.4$


smaller than 2
larger than 2/3
2 − 8/5
3.3 − 2.9
1/3 × 2/3
3 ÷ 4
3/2 − 3/4
3 × 0.4
7/10 + 1/2
1.8 + 0.4
5 × 0.5
3/2 × 5/3

Pair the tags.


7/10 − 1/5
1 3/10 − 7/10
1/4 + 3/10
0.7 × 0.9
7 × 0.9
0.7 × 0.09
0.55
6.3
0.063
0.5
0.63
0.6

Complete.

$120^2 =$ 14,400

$1,200^2 =$ 1,440,000

$12,000^2 =$ 144,000,000

$\boxed{700}^2 = 490,000$

$\boxed{40}^3 = 64,000$

$30^3 =$ 27,000

$300^3 =$ 27,000,000

$3,000^3 =$ 27,000,000,000

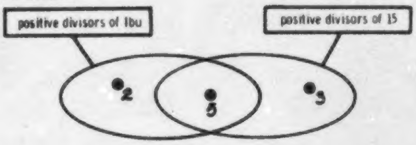
$\boxed{60}^2 = 3,600$

$\boxed{100}^3 = 1,000,000$

First clue

Ibu is a whole number smaller than 100.

Second clue


positive divisors of Ibu
positive divisors of 15
2
5
3

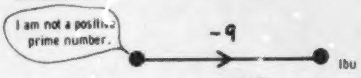
Ibu could be 10, 20, 40, 50, 70, or 80.

Third clue

Ibu =, 16

Ibu could be 10, 40, or 70.

Fourth clue


I am not a positive prime number.
-9
Ibu

Ibu is 40.

Each table is for one of these:

| + |
|---|
| × |
| $+_{10}$ |
| $\times_5$ |
| $T_D$ |
| $T_a$ |

Label the tables.

| × | 4 | 9 |
|---|---|---|
| 0 | 0 | 0 |
| 6 | 24 | 54 |

| $+_{10}$ | 6 | 3 |
|---|---|---|
| 4 | 0 | 7 |
| 8 | 4 | 1 |

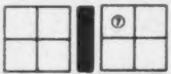
| $T_D$ | 8 | 30 |
|---|---|---|
| 5 | 0 | 1 |
| 2 | 1 | 1 |

| $\times_5$ | 1 | 5 |
|---|---|---|
| 6 | 1 | 0 |
| 0 | 0 | 0 |

Elsa is a secret number.

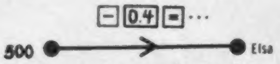
First clue

Elsa can be put on this Minicomputer by adding exactly one ⑨-checker.



Elsa could be 6.5, 7.4, 9.2, 12.8, 14.6, 23.6, 41.6, or 77.6.

Second clue



Elsa could be 9.2, 12.8, 23.6, 41.6, or 77.6.

Third clue

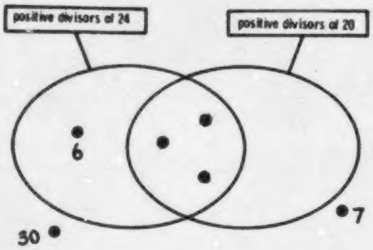
A name for Elsa can be written by adding parentheses to this expression.

$$8 \times (4 + (6 \div 5))$$

Who is Elsa? 41.6

| The red tag is one of these: | The blue tag is one of these: |
|---|---|
| multiples of 3 | multiples of 3 |
| multiples of 4 | multiples of 4 |
| multiples of 5 | multiples of 5 |
| odd numbers | odd numbers |
| positive prime numbers | positive prime numbers |
| smaller than 10 | smaller than 10 |
| positive divisors of 18 | positive divisors of 18 |
| positive divisors of 20 | positive divisors of 20 |
| positive divisors of 24 | positive divisors of 24 |

Label the strings.



First clue

Tip is a secret whole number between 20 and 30.
Pip is a secret whole number between 30 and 40.

Second clue

| $=_3$ | Pip | 10 |
|---|---|---|
| Tip | False | True |
| 21 | False | False |

Tip could be 22, 25, or 28.

Pip could be 32, 35, or 38.

Third clue

$$\text{Tip} \sqcap \text{Pip} = 2$$

(Tip, Pip) could be (22, 32), (22, 38), or (28, 38).

Fourth clue

$$\text{Tip} =_4 \text{Pip}$$

(Tip, Pip) is (22, 38).

Loki is a secret whole number.

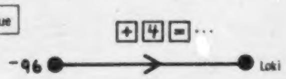
First clue

$$\text{Loki} \sqcap 15 = 5$$

Find a pattern for the numbers that Loki could be.

Loki could be 5 or 10 or 20 or 25 or 35 or 40 or 50 or 55, and so on. (multiples of 5 which are not multiples of 3)

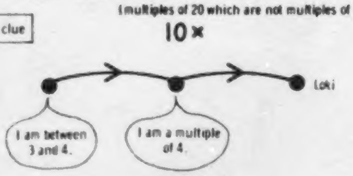
Second clue



Find a pattern for the numbers Loki could be.

Loki could be 20 or 40 or 80 or 100 or 140, and so on.
(multiples of 20 which are not multiples of 3)

Third clue



Loki is 320.

# ARCADE OF PROBLEMS

# #5

# ARCADE OF PROBLEMS #5

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.* You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems.

A brief collective lesson is suggested to begin the second workbook lesson. This should be a brief warm-up activity to review the "a * b" format.

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* See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

# W 11&12

## Lesson 1

Give each student a copy of "Arcade of Problems #5" and let them work on their own.

If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

## Lesson 2

Write the following information on the board.

$$\boxed{a * b = (2 \times a) + b}$$

$5 * 8 =$ $9 * 7 =$

$8 * 5 =$ $7 * 9 =$

Call the class' attention to what is written in the red box.

T: This is the rule for the operation ⋆. It tells us how to calculate "a ⋆ b" (read: a star b) for any numbers a and b. What number is 5 ⋆ 8?

S: 18.

Do not ask for an explanation of the calculation yet. Invite students first to complete all four calculations.

$5 * 8 = 18$ $9 * 7 = 25$

$8 * 5 = 21$ $7 * 9 = 23$

Point to the rule in the red box.

T: Who can explain this rule?

S: You multiply the first number (a) by 2 and then add on the second number (b). For example, 5 * 8 = 18 because 2 x 5 = 10 and 10 + 8 = 18.

T: Who can explain why 8 * 5 = 21?

S: 8 * 5 = (2 x 8) + 5 = 16 + 5 = 21.

After the rule has been explained, write the following calculations on the board. Invite students to complete them and to explain their solutions. Answers are in the boxes.

$$7 * 7 = \boxed{21} \qquad 0 * 11 = \boxed{11}$$

$$7 * {}^{-}5 = \boxed{9} \qquad 0.8 * 0.6 = \boxed{2.2}$$

Return the students' copies of "Arcade of Problems #5". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

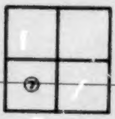
At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

# Arcade of Problems # 5

Lipa is a secret number.

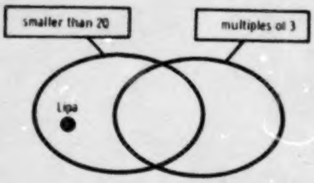
**First clue**

Lipa can be shown on this Minicomputer by adding exactly one regular positive checker: ●



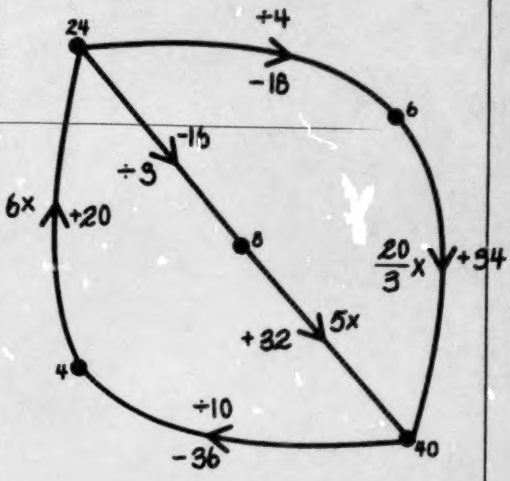
Lipa could be 15, 16, 18, or 22.

**Second clue**


smaller than 20
multiples of 3
Lipa

Lipa is 16

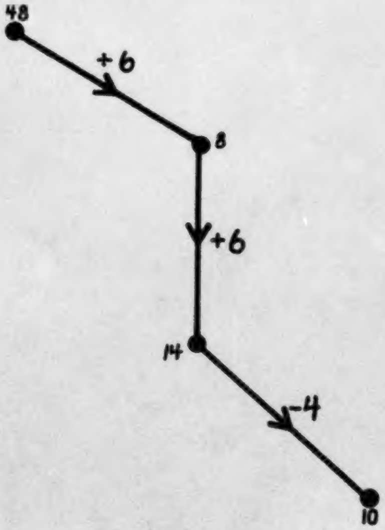
Label the arrows: Try to label some arrows in two ways. Many solutions are possible.


24
+4
−18
6
−15
÷3
6x
+20
8
20/3 x
+34
5x
+32
4
÷10
−36
40

(Other solutions are possible.)

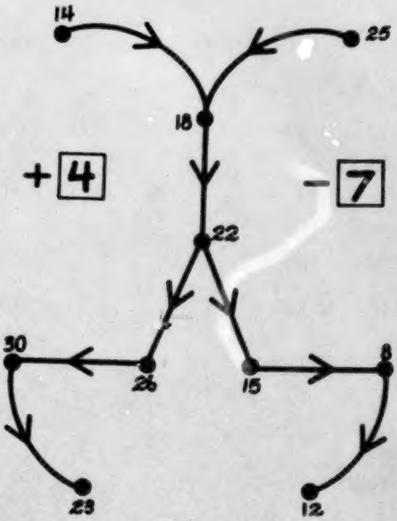
Build an arrow road from 48 to 10. Each arrow must be for one of these:

+4, +6, −4, −6, x4, x6, ÷4, ÷6


48
÷6
8
+6
14
−4
10

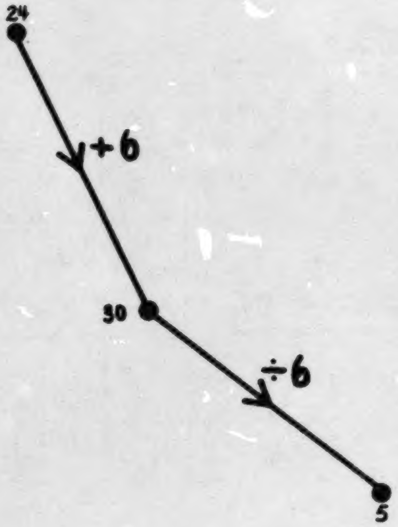
(Many solutions are possible. The shortest roads have three arrows.)

Fill in the boxes for the arrows and label the dots.


14
25
18
+4
−7
22
30
26
15
8
23
12

Build an arrow road from 24 to 5. Each arrow must be for one of these:

+4, +6, -4, -6, x4, x6, ÷4, ÷6



(Many solutions are possible. The shortest roads have two arrows.)

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Add

$$\begin{array}{r} \boxed{5}\,8 \\ 7\,\boxed{5}\,0\,5 \\ +\,8\,9\,2\,\boxed{4} \\ \hline \boxed{1}\,\boxed{6}\,4\,8\,7 \end{array}$$

Add

375 + 5092 + 28 + 6

5,501

Subtract

$$\begin{array}{r} \boxed{7}\,5\,3 \\ -\,2\,\boxed{6}\,6 \\ \hline 4\,8\,\boxed{7} \end{array}$$

Subtract

25,027 - 793

24,234

Cho is a secret whole number.

**First clue**

Cho is one of these numbers.

$\sqcap$ : Greatest common divisor

$\sqcup$ : Least common multiple

$21 \sqcap 6 = \underline{3}$

$20 \sqcap 9 = \underline{1}$

$14 \sqcap 35 = \underline{7}$

$10 \sqcup 4 = \underline{20}$

$3 \sqcup 7 = \underline{21}$

$9 \sqcup 6 = \underline{18}$

**Second clue**

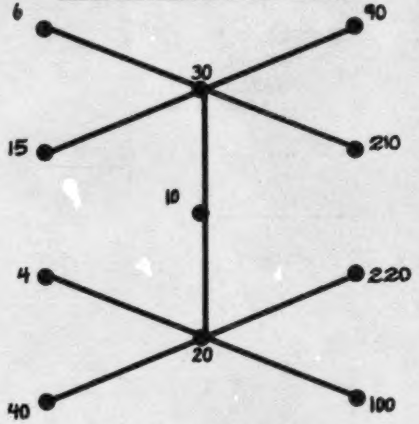
Cho =, 11

Who is Cho? 20

Label the dots. Many solutions are possible.

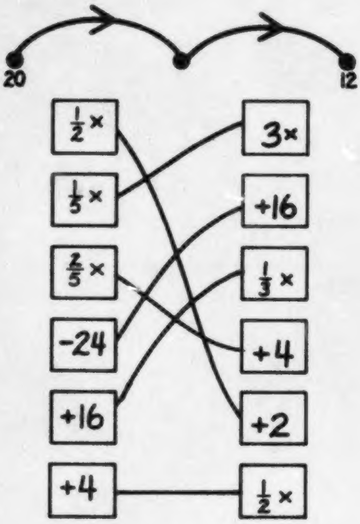
**Prime Factor Relation**

Two whole numbers are joined by a red cord if and only if one of the numbers equals the other number times a positive prime number.



(Many solutions are possible. The center dot must be for 10 or 60.)

Pair the tags.


20
12
$\frac{1}{2}\times$
$\frac{1}{5}\times$
$\frac{2}{5}\times$
-24
+16
+4
3×
+16
$\frac{1}{3}\times$
+4
+2
$\frac{1}{2}\times$

10

Complete each of these tables.

| $-$ | 15 | 26 |
|---|---|---|
| 23 | 8 | $^{-}3$ |
| 12 | $^{-}3$ | $^{-}14$ |

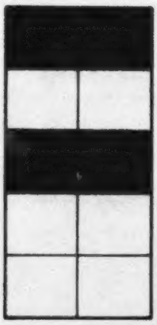
| $+_5$ | 6 | 12 |
|---|---|---|
| 2 | 3 | 4 |
| 8 | 4 | 0 |

| $\times_5$ | 4 | 11 |
|---|---|---|
| 2 | 2 | 1 |
| 6 | 0 | 0 |

| $T_>$ | 2 | $^{-}50$ |
|---|---|---|
| 27 | 1 | 1 |
| $^{-}5$ | 0 | 1 |

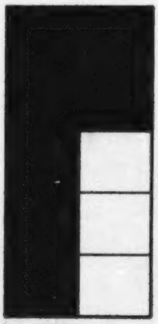
11

Color two-fifths of this region red and fill in the box.



$$\frac{2}{5} = \frac{\boxed{4}}{10}$$

Color seven-tenths of this region red.



$$\frac{7}{10}$$

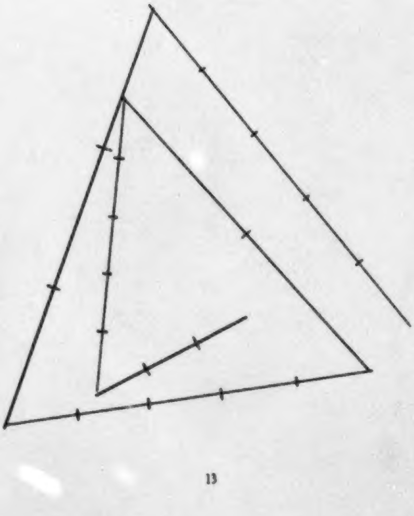
Complete.

$$\frac{2}{5} + \frac{7}{10} = \frac{11}{10} = 1\frac{1}{10}$$

12

(Other colorings are possible.)

Using a ruler, divide the red segment into two pieces of equal length.
Divide each blue segment into three pieces of equal length.
Divide each black segment into five pieces of equal length.
One black segment is done for you.



13

$\boxed{a * b = (2 \times a) + b}$

Example: $5 * 3 = (2 \times 5) + 3 = 13$

Complete.

$7 * 4 = \boxed{18}$ $4 * 7 = \boxed{15}$

$^{-}8 * 10 = \boxed{^{-}6}$ $35 * {}^{-}4 = \boxed{66}$

$6 * \boxed{3} = 15$ $19 * \boxed{^{-}3} = 35$

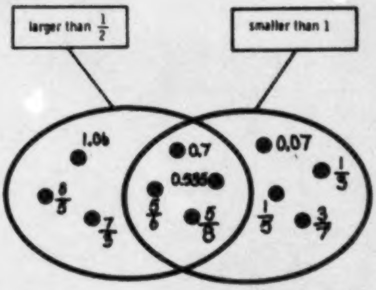
$\boxed{7} * 4 = 18$ $\boxed{13} * {}^{-}5 = 21$

Put each of these numbers in the string picture.

0.07 0.7 1.06 0.535

$\frac{1}{3}$ $\frac{1}{5}$ $\frac{5}{6}$

$\frac{5}{8}$ $\frac{3}{7}$ $\frac{8}{5}$ $\frac{7}{3}$


larger than $\frac{1}{2}$
smaller than 1
1.06
$\frac{8}{5}$
$\frac{7}{3}$
0.7
0.535
$\frac{5}{6}$
$\frac{5}{8}$
0.07
$\frac{1}{5}$
$\frac{1}{3}$
$\frac{3}{7}$

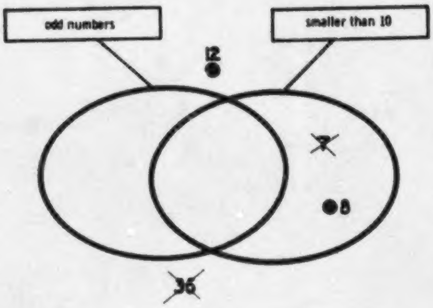
The red tag is one of these:

- multiples of 2
- multiples of 3
- multiples of 5
- odd numbers
- positive prime numbers
- smaller than 10
- positive divisors of 24

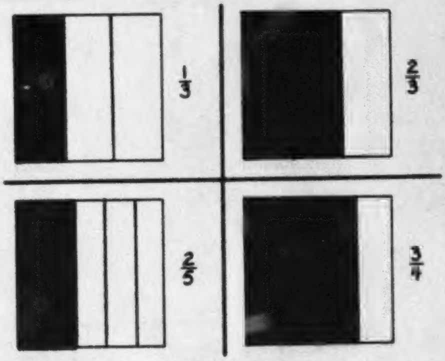
The blue tag is one of these:

- multiples of 2
- multiples of 3
- multiples of 5
- odd numbers
- positive prime numbers
- smaller than 10
- positive divisors of 24

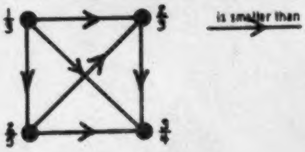
Label the strings.


odd numbers
smaller than 10
12
7
8
36

Color in red the indicated fractional part of each square. Use a ruler to divide the squares.


$\frac{1}{3}$
$\frac{2}{3}$
$\frac{2}{5}$
$\frac{3}{4}$

Draw all the missing red arrows between these dots.


$\frac{1}{3}$
$\frac{2}{3}$
$\frac{2}{5}$
$\frac{3}{4}$
is smaller than

(Other colorings are possible.)

$\boxed{a * b = (2 \times a) + b}$

**Example:** $0.7 * 0.3 = (2 \times 0.7) + 0.3 = 1.4 + 0.3 = 1.7$

**Complete.**

$0.3 * 0.2 = \boxed{0.8}$ $0.6 * 0.5 = \boxed{1.7}$

$1.2 * 0.3 = \boxed{2.7}$ $0.9 * 0.4 = \boxed{2.2}$

$0.4 * \boxed{0.5} = 1.3$ $2.1 * \boxed{0.7} = 4.9$

$\boxed{0.3} * 0.2 = 0.8$ $\boxed{1.4} * 1.0 = 3.8$

---

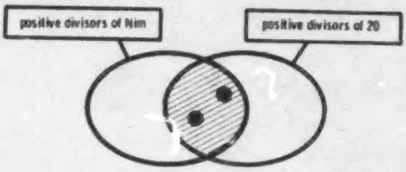
Nim is a secret whole number.

First clue

Nim $\square$ 10 = 30

Nim could be 3, 6, 15, or 30.

Second clue



Nim could be 6 or 15.

Third clue

Nim =, 27

Nim is 15.

---

Each table is for one of these:

| | | |
|---|---|---|
| $+$ | $+_3$ | $+_5$ |
| $-$ | $-_3$ | $-_5$ |
| $\times$ | $\times_3$ | $\times_5$ |
| $T_=$ | $T_<$ | $T_>$ |
| | ↑ | ↓ |

Label the tables.

| $+_3$ | 3 | 1 |
|---|---|---|
| 6 | 0 | 1 |
| 7 | 1 | 2 |

| $\times_6$ | 3 | 6 |
|---|---|---|
| 6 | 3 | 1 |
| 5 | 0 | 0 |

| $T_=$ | 21 | 12 |
|---|---|---|
| 7 | 1 | 0 |
| 10 | 0 | 1 |

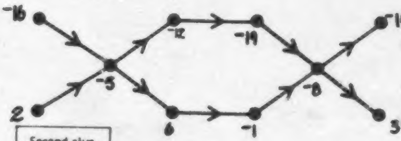
| ↓ | 4 | 10 |
|---|---|---|
| 1 | 1 | 1 |
| 7 | 4 | 7 |

---

Sol is a secret number.

First clue

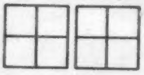
Sol is in this arrow picture.



−7
+11

Second clue

Sol can be put on this Minicomputer with exactly these two checkers: ③, Ⓐ.



Sol could be ⁻16, ⁻8, ⁻5, ⁻1, or 2.

Third clue

A name for Sol can be written using exactly these symbols.

4 × ) 7 9 ( −

$4 \times (7-9) = {}^-8$

Who is Sol? ⁻8

Fill in each box with a whole number.

Example: $2^{\boxed{5}} = 2 \times 2 \times 2 \times 2 \times 2 = 32$

$8^3 = \boxed{512}$ $30^2 = \boxed{900}$

$8^4 = \boxed{4{,}096}$ $30^3 = \boxed{27{,}000}$

$7^{\boxed{2}} = 49$ $\boxed{9}^2 = 81$

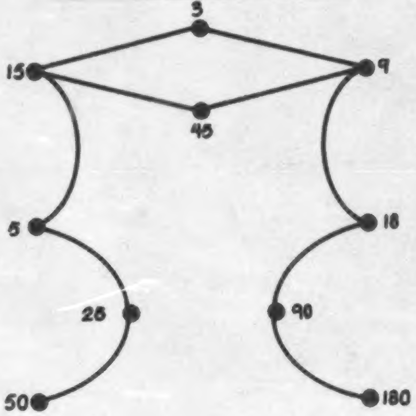
$5^{\boxed{3}} = 125$ $\boxed{6}^3 = 216$

$3^{\boxed{5}} = 243$ $\boxed{50}^2 = 2{,}500$

Label the dots. Many solutions are possible.

**Prime Factor Relation**

Two whole numbers are joined by a red cord if and only if one of the numbers equals the other number times a positive prime number.



(Many solutions are possible.)

Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Multiply

$$\begin{array}{r} 809 \\ \times\, 607 \\ \hline 491{,}063 \end{array}$$

Multiply

$$\begin{array}{r} 7\boxed{8} \\ \times\, \boxed{7}4 \\ \hline \boxed{3}12 \\ +\,5\boxed{4}\boxed{6}\phantom{0} \\ \hline \boxed{5}\boxed{7}7\boxed{2} \end{array}$$

Divide

$$\begin{array}{rl} \boxed{3}.\boxed{2}\boxed{4} \text{ R-}0.\boxed{1}\boxed{3} & \\ \boxed{3}2\overline{)\,103.\boxed{8}1} & \\ -\boxed{9}\boxed{6}.\boxed{0}\boxed{0} & 3.00 \\ \hline \boxed{7}.\boxed{8}\boxed{1} & \\ -\boxed{6}.\boxed{4}\boxed{0} & .\boxed{2}0 \\ \hline \boxed{1}.4\boxed{1} & \\ -\boxed{1}.\boxed{2}\boxed{8} & .0\boxed{4} \\ \hline .\boxed{1}\boxed{3} & \end{array}$$

Divide

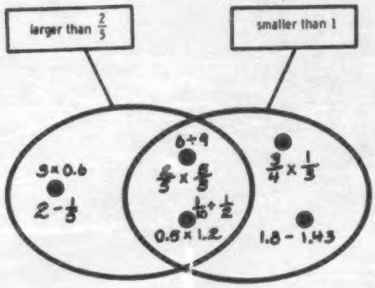
$$\begin{array}{r} \boxed{4}.\boxed{1}\boxed{5} \text{ R-}0.\boxed{2}\boxed{2} \\ 29\overline{)\,120.57} \end{array}$$

Label the dots. Some of the numbers have two names listed. Write both names for those numbers beside their dots.

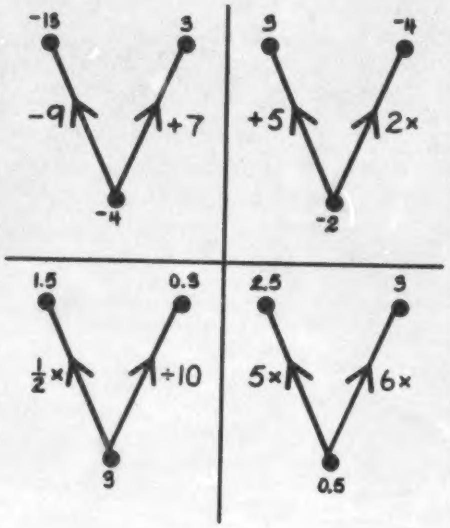
$6 \div 9$ $3 \times 0.6$ $0.5 \times 1.2$

$1.8 - 1.43$ $\frac{1}{10} + \frac{1}{2}$

$2 - \frac{1}{5}$ $\frac{3}{4} \times \frac{1}{3}$ $\frac{2}{5} \times \frac{5}{3}$


larger than $\frac{2}{5}$
smaller than 1
3×0.6
2−$\frac{1}{5}$
6÷9
$\frac{2}{5}$×$\frac{5}{3}$
$\frac{1}{10}$+$\frac{1}{2}$
0.5×1.2
$\frac{3}{4}$×$\frac{1}{3}$
1.8 − 1.43

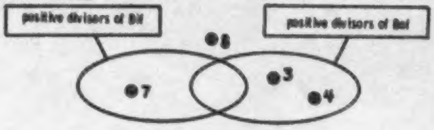
3 is the largest number in each picture. Label the dots.


-15
3
-9
+7
-4
3
-4
+5
2×
-2
1.5
0.3
½×
÷10
3
2.5
3
5×
6×
0.5

**First clue**

Bif and Bof are secret whole numbers.
Bif is smaller than 50 and Bof is smaller than 100.

**Second clue**


positive divisors of Bif
positive divisors of Bof
8
7
3
4

Bif could be 7, 14, 35, or 49.
Bof could be 12, 36, or 60.

**Third clue**

Bif + Bof = 0

(Bif, Bof) could be (14, 36), (35, 60), or (49, 36).

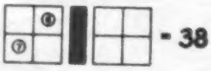
**Fourth clue**

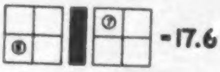
Bif ⊓ Bof = 1

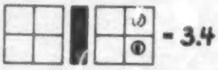
(Bif, Bof) is (49, 36).

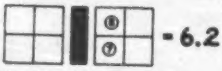
Put each number on the Minicomputer with exactly these two checkers.

⑥ ⑦

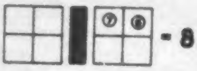


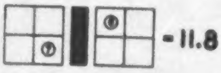






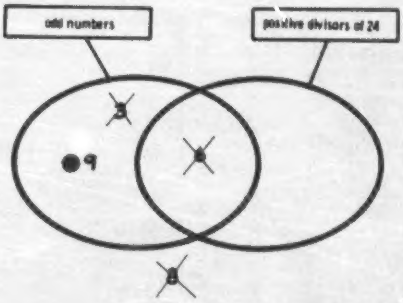
(There is another solution with
⑦ on the 0.8 - square and
⑥ on the 0.1 - square.)



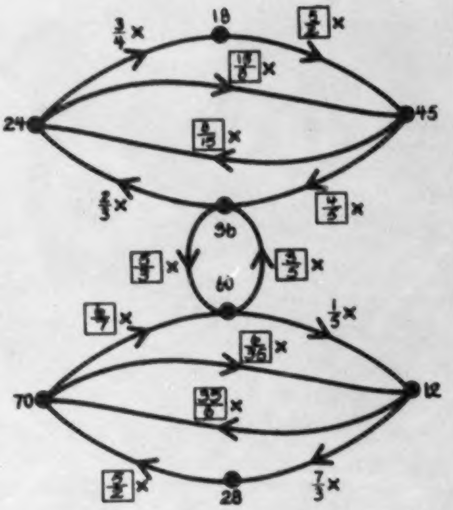


| The red tag is one of these: | The blue tag is one of these: |
|---|---|
| multiples of 4 | multiples of 4 |
| odd numbers | odd numbers |
| smaller than 10 | smaller than 10 |
| positive divisors of 12 | positive divisors of 12 |
| positive divisors of 18 | positive divisors of 18 |
| positive divisors of 24 | positive divisors of 24 |

Label the strings.


odd numbers
positive divisors of 24
3
9
1
8

Label the dots and fill in the boxes for the arrows.

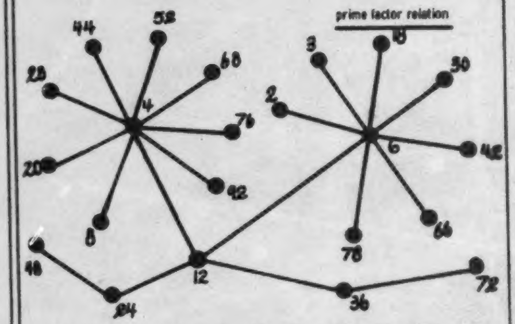

$\frac{3}{4}$×
18
$\boxed{\frac{5}{2}}$×
$\boxed{\frac{15}{8}}$×
24
45
$\boxed{\frac{8}{15}}$×
$\frac{2}{3}$×
$\boxed{\frac{4}{5}}$×
36
$\boxed{\frac{5}{3}}$×
$\boxed{\frac{3}{5}}$×
60
$\boxed{\frac{6}{7}}$×
$\frac{1}{5}$×
$\boxed{\frac{6}{35}}$×
70
12
$\boxed{\frac{35}{6}}$×
$\boxed{\frac{5}{2}}$×
28
$\frac{7}{3}$×

30

Prif is a whole number smaller than 100.

The prime factor distance from 12 to Prif is 2.

pfd (12, Prif) = 2


prime factor relation
44
52
68
28
4
76
20
92
8
48
24
12
3
18
30
2
6
42
78
66
36
72

Prif could be 2, 3, 8, 18, 20, 28, 30, 42, 44, 48, 52, 66, 68, 72, 76, 78, or 92.

31

(Other cord pictures could show the same solutions.)

| | | | | |
|---|---|---|---|---|
| 66 | 1 | 20 | 512 | 1,000 |
| | $\frac{3}{10}$ | $\frac{8}{9}$ | $\frac{11}{3}$ | 0.25 |
| 0 | 0.2 | 1.5 | $^-5$ | $^-12$ |

Names for the above numbers can be written using just these symbols:

2 3 + − × ÷ ( )

For each number, use "2" and "3" exactly twice and the other symbols as often as you wish.

Example: $2^{(3+3)}+2 = 2^6+2 = 64+2 = 66$

Write names for eight of the above numbers.

$3^2-2^3=1$ $(3\times2\times3)+2=20$ $2^{(3\times2)+3}=512$

$((2+3)\times2)^3=1{,}000$ $3\div(2\times(3+2))=\frac{3}{10}$ $2^3\div3^2=\frac{8}{9}$

$(3^2+2)\div3=\frac{11}{3}$ $(3\div3)\div2^2=\frac{1}{4}=0.25$ $(3-2)\div(3+2)=\frac{1}{5}=0.2$

$((3\times2)-3)\div2=\frac{3}{2}=1.5$ $2^3-2^3=0$ $(3-2)-(3\times2)={}^-5$

$3\times(2-(2\times3))={}^-12$

# ARCADE
# OF
# PROBLEMS
# #6

## ARCADE OF PROBLEMS #6

The problems in this workbook are in order of increasing difficulty. Approximately the first ten pages of the workbook have easy problems, the next ten to twelve pages have average problems, and the last ten pages have more challenging problems. It is suggested that all students work through as many pages as they can handle during the two lessons scheduled for the workbook. Not all students are expected to complete the entire workbook; most should complete at least ten pages, many should finish about twenty pages, and only the faster students will reach the most challenging problems at the end of the booklet.

Many problems in the workbook were designed with the availability of hand-calculators in mind. If possible, provide all the students with hand-calculators or at least give students access to hand-calculators.* You will find that many students do not need or want to use calculators; however, some students still need the support of the calculator in thinking about these problems.

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* See Section 2(f) of the General Introduction for additional comments on the use of hand-calculators and on desirable characteristics of hand-calculators.

# W 14 & 15

## Lesson 1

Give each student a copy of "Arcade of Problems #6" and let them work on their own.

If you observe many students struggling with the same problem (or part of a problem), you could have a short collective discussion about that problem.

Collect the workbooks at the end of the lesson. Check the students' work and have the workbooks ready to return for Lesson 2.

## Lesson 2

Return the students' copies of "Arcade of Problems #6". You may wish to discuss any page that was difficult for most of the students. Then ask the students to correct their mistakes, if any, and to continue working in their workbooks.

If you observe many students struggling with the same problem (or part of a problem), you could have a short discussion of that problem.

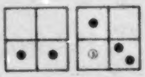
At the end of this lesson, collect the workbooks for your progress check. After checking the workbooks, you may wish to ask some students to work further during any free study time or as a homework assignment.

# Arcade of Problems # 6

Yip is a secret number.

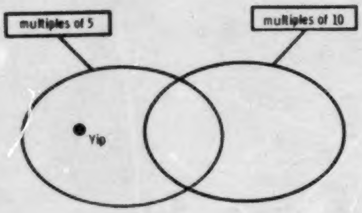
**First clue**

Yip can be shown on this Minicomputer by taking off exactly one checker.



Yip could be 26 , 36 , 38 , 40 , or 45 .

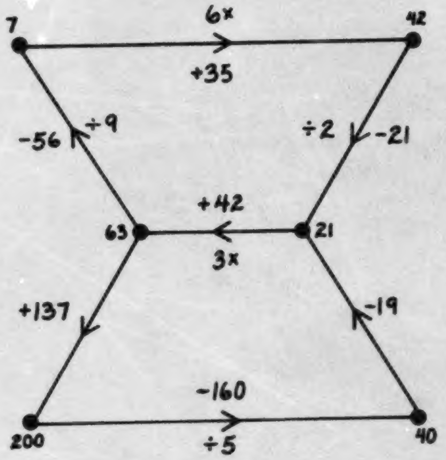
**Second clue**


multiples of 5
multiples of 10
Yip

Who is Yip? 45

2

Label the red arrows; try to label some arrows in two ways.
Many solutions are possible.

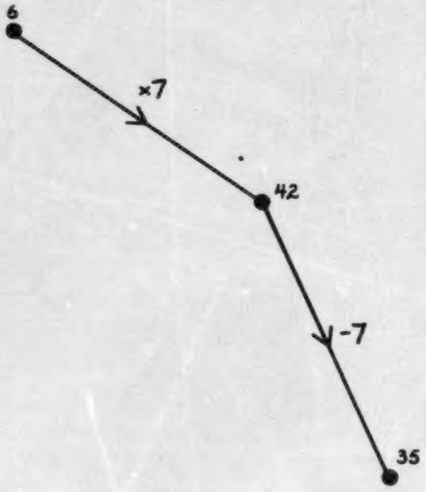

7
6×
+35
42
-56
÷9
÷2
-21
+42
63
21
3×
+137
-19
-160
200
÷5
40

3

(Other solutions are possible.)

Build an arrow road from 6 to 35. Each arrow must be for one of these:

+2, +7, -2, -7, x2, x7, ÷2, ÷7



4

(Many solutions are possible. The shortest roads have two arrows.)

Nama is a secret number.

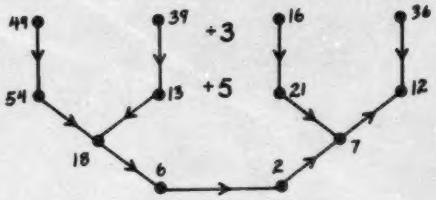
**First clue**

Nama is larger than 30 and can be put on this Minicomputer with exactly these two checkers: ● and ④ .

Nama could be 32 , 49 , 50 , 52 , or 56 .

**Second clue**

Nama is in this arrow picture.

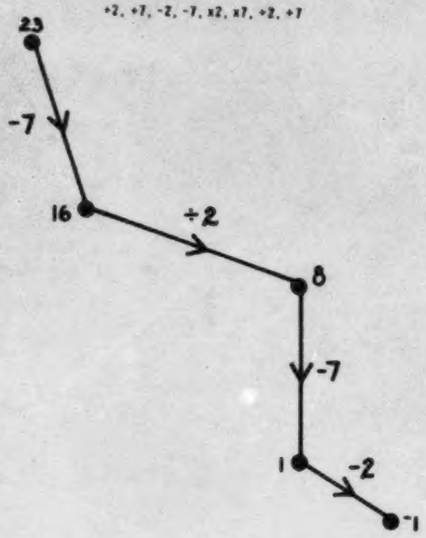

49
39
÷3
16
36
54
13
+5
21
12
18
7
6
2

Who is Nama? 49

5

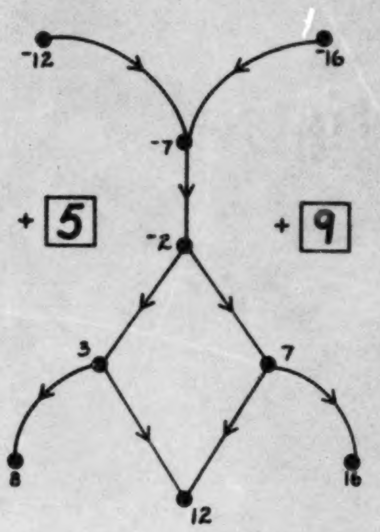
Build an arrow road from 23 to ⁻1. Each arrow must be for one of these:

+2, +7, -2, -7, x2, x7, ÷2, ÷7



(Many solutions are possible. The shortest roads have four arrows.)

Fill in the boxes for the arrows and label the dots.



Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

Add

$$\begin{array}{r} \boxed{5}\,5\,6 \\ 6\,2\,\boxed{5}\,7 \\ +\quad 8\,3 \\ \hline \boxed{6}\,8\,9\,\boxed{6} \end{array}$$

Add

58 + 3,009 + 270 + 1,068

4,405

Subtract

$$\begin{array}{r} 4\,2\,6\,\boxed{9} \\ -\,\boxed{3}\,7\,3 \\ \hline \boxed{3}\,8\,\boxed{9}\,6 \end{array}$$

Subtract

$$\begin{array}{r} 5\,0\,8\,\boxed{3} \\ -\,\boxed{3}\,4\,\boxed{7}\,9 \\ \hline 1\,\boxed{6}\,0\,4 \end{array}$$

Han is a secret number.

First clue

Han is one of these numbers.

⊓ : Greatest common divisor ⊔ : Least common multiple

$8 \sqcap 28 =$ 4

$24 \sqcap 15 =$ 3

$12 \sqcap 36 =$ 12

$10 \sqcup 6 =$ 30

$15 \sqcup 9 =$ 45

$7 \sqcup 4 =$ 28

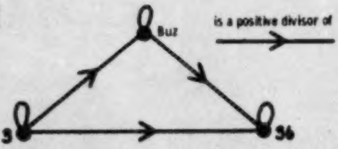
Second clue

$\text{Han} \equiv_4 18$

Who is Han? 30

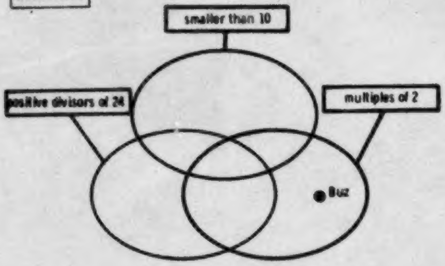
Buz is a secret number.

First clue


Buz
is a positive divisor of
3
36

Buz could be 6, 9, 12, or 18.

Second clue


smaller than 10
positive divisors of 24
multiples of 2
Buz

Buz is 18.

Complete each of these tables.

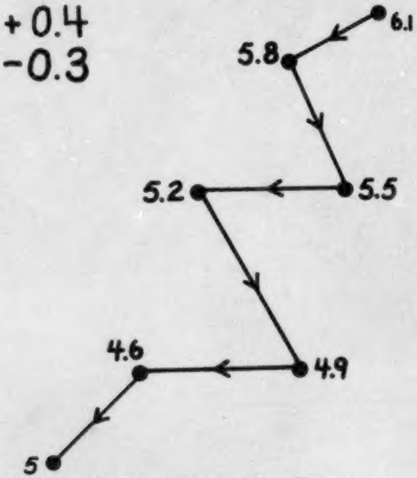
| $+_5$ | 8 | 24 |
|---|---|---|
| 5 | 3 | 4 |
| 37 | 0 | 1 |

| $\times_3$ | 5 | 7 |
|---|---|---|
| 4 | 2 | 1 |
| 6 | 0 | 0 |

| ↓ | 23 | 0 |
|---|---|---|
| 7 | 7 | 0 |
| $^{-}6$ | $^{-}6$ | $^{-}6$ |

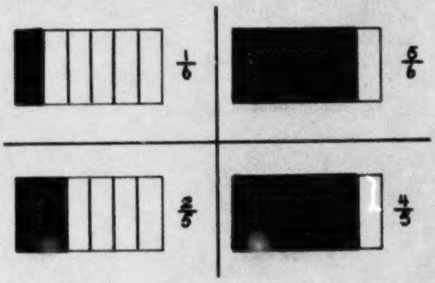
| $T_D$ | 72 | 42 |
|---|---|---|
| 7 | 0 | 1 |
| 12 | 1 | 0 |

Build a red-blue road between 5 and 6.1.

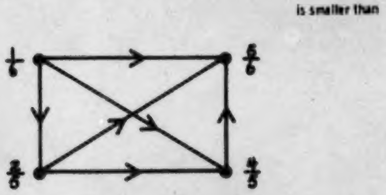

+0.4
-0.3
6.1
5.8
5.5
5.2
4.9
4.6
5

(Many solutions are possible. The shortest roads have five red arrows and one blue arrow.)

Color in red the indicated fractional part of each rectangle. Use a ruler to divide the rectangles.


$\frac{1}{6}$
$\frac{5}{6}$
$\frac{2}{5}$
$\frac{4}{5}$

Draw all the missing red arrows between these dots.


is smaller than
$\frac{1}{6}$
$\frac{5}{6}$
$\frac{2}{5}$
$\frac{4}{5}$

$$\boxed{a * b = (2 \times a) - b}$$

Example: $3 * 7 = (2 \times 3) - 7 = {}^{-}1$

Complete.

$9 * 7 = \boxed{11}$ $7 * 9 = \boxed{5}$

$13 * 11 = \boxed{15}$ $3 * 8 = \boxed{{}^{-}2}$

$0 * 6 = \boxed{{}^{-}6}$ $5 * 23 = \boxed{{}^{-}13}$

$12 * \boxed{5} = 19$ $5 * \boxed{12} = {}^{-}2$

$\boxed{8} * 3 = 13$ $\boxed{4} * 7 = 1$

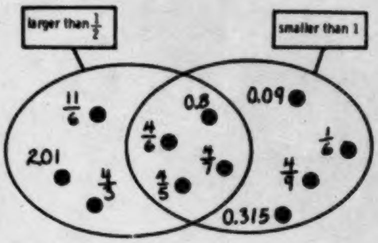
---

Put these numbers in the string picture.

0.09 2.01 0.8 0.315

$\frac{1}{6}$ $\frac{4}{6}$ $\frac{11}{6}$

$\frac{4}{3}$ $\frac{4}{5}$ $\frac{4}{7}$ $\frac{4}{9}$

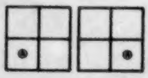

larger than $\frac{1}{2}$
smaller than 1
$\frac{11}{6}$
2.01
$\frac{4}{3}$
0.8
$\frac{4}{6}$
$\frac{4}{7}$
$\frac{4}{5}$
0.09
$\frac{1}{6}$
$\frac{4}{9}$
0.315

---

Com is a secret number.

**First clue**

Com can be put on this Minicomputer by adding exactly one ③-checker.



Com could be ${}^{-}18$, ${}^{-}15$, ${}^{-}9$, 3, 9, 39, 99, or 219.

**Second clue**

$\text{Com} =_4 3$

Com could be ${}^{-}9$, 3, 39, 99, or 219.

**Third clue**

A name for Com can be written by adding parentheses to this expression.

$8 - ((4 \times 3) + 5)$

Who is Com? ${}^{-}9$

---

Each table is for one of these:

| | |
|---|---|
| $+$ | $+_2$ |
| $\times$ | $\times_5$ |
| $+_5$ | $+_{10}$ |
| $\times_3$ | $\times_{10}$ |
| ↑ | ↓ |
| $T_D$ | $T_n$ |
| $T_{=2}$ | |

Label the tables.

| $+_5$ | 6 | 9 |
|---|---|---|
| 8 | 4 | 2 |
| 11 | 2 | 0 |

| $T_n$ | 6 | 24 |
|---|---|---|
| 30 | 1 | 0 |
| 12 | 1 | 0 |

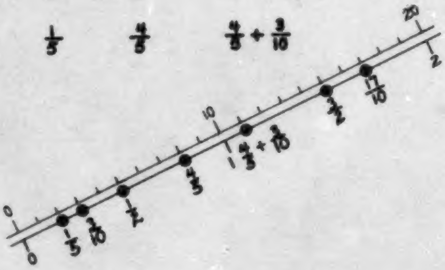
| ↑ | 8 | 1 |
|---|---|---|
| 0 | 8 | 1 |
| 13 | 13 | 13 |

| $\times_3$ | 4 | 11 |
|---|---|---|
| 6 | 0 | 0 |
| 7 | 1 | 2 |

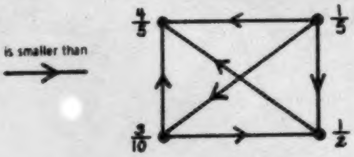
Label the dots on the number line with these numbers.

$\frac{1}{2}$ $\frac{3}{2}$ $\frac{3}{10}$ $\frac{17}{10}$

$\frac{1}{5}$ $\frac{4}{5}$ $\frac{4}{5} + \frac{3}{10}$


0
10
20
0
1
2
1/5
3/10
1/2
4/5
1 4/5 + 3/10
3/2
17/10

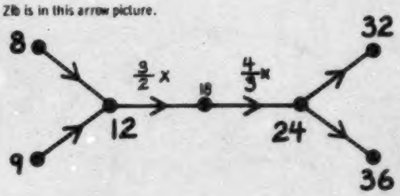
Draw all the missing red arrows between these dots.


is smaller than
4/5
1/5
3/10
1/2

Zib is a secret number.

First clue

Zib is in this arrow picture.


8
9
12
3/2 x
18
4/3 x
24
32
36

Second clue

$$\text{Zib} \sqcap 40 = 4$$

Zib could be 12 or 36.

Third clue

$$20 \uparrow \text{Zib} = \text{Zib}$$

Zib is 36.

$$a * b = (2 \times a) - b$$

Complete.

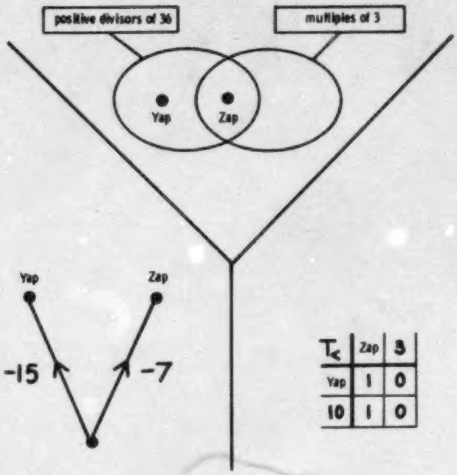
$0.4 * 0.5 = \boxed{0.3}$ $0.7 * 1.2 = \boxed{0.2}$

$1.8 * 1.8 = \boxed{1.8}$ $1.5 * 3.4 = \boxed{^{-}0.4}$

$2.6 * \boxed{1.4} = 3.8$ $0.8 * \boxed{1.9} = {}^{-}0.3$

$\boxed{0.7} * 0.9 = 0.5$ $\boxed{0.8} * 2 = {}^{-}0.4$

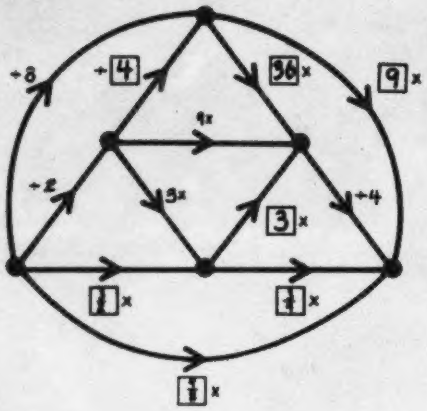
Yap and Zap are secret whole numbers.


positive divisors of 36
multiples of 3
Yap
Zap
Yap
Zap
-15
-7

| T< | Zap | 3 |
|---|---|---|
| Yap | 1 | 0 |
| 10 | 1 | 0 |

(Yap, Zap) = (4, 12)

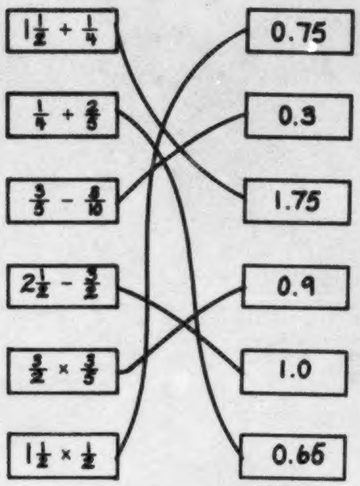
Fill in the boxes for the arrows.



÷8
÷4
36 x
9 x
9x
÷2
5x
3 x
÷4
x
x
x

22

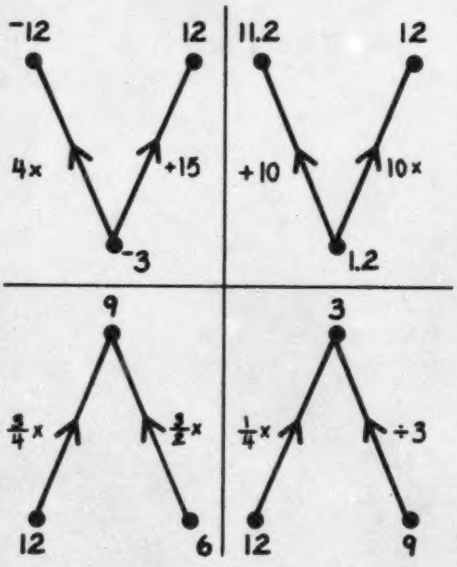
Pair the tags.



1½ + ¼
¼ + 2/5
3/5 − 8/10
2½ − 3/2
3/2 × 3/5
1½ × ½
0.75
0.3
1.75
0.9
1.0
0.65

23

12 is the largest number in each picture. Label the dots.



−12
12
4x
+15
−3
11.2
12
+10
10x
1.2
9
3/4 x
3/2 x
12
6
3
¼ x
÷3
12
9

24

Each table is for one of these:

| | | | |
|---|---|---|---|
| $+$ | $+_3$ | $+_5$ | $+_{10}$ |
| $-$ | $-_3$ | $-_5$ | $-_{10}$ |
| $\times$ | $\times_3$ | $\times_5$ | $\times_{10}$ |
| ↓ | $T_{=_3}$ | $T_<$ | $\sqcap$ |
| ↑ | $T_{=_4}$ | $T_>$ | $\sqcup$ |
| | $T_=$ | $T_{\neq}$ | |

Label the tables.

| $\times_5$ | 8 | 15 |
|---|---|---|
| 4 | 2 | 0 |
| 7 | 1 | 0 |

| $T_{=_4}$ | 10 | 11 |
|---|---|---|
| 5 | 0 | 0 |
| 2 | 1 | 0 |

| $-_{10}$ | 7 | 13 |
|---|---|---|
| 11 | 4 | 8 |
| 19 | 2 | 6 |

| ↓ | 10 | 2 |
|---|---|---|
| 1 | 1 | 1 |
| 4 | 4 | 2 |

25

Put exactly one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box.

Do = 3,87☐ and Do is divisible by 2.

Do could be 3,870, 3,872, 3,874, 3,876, or 3,878.

---

Re = 7,☐23 and Re is divisible by 9.

Re = 7,623.

---

Mi = 5,4☐6 and Mi is divisible by 4.

Mi could be 5,416, 5,436, 5,456, 5,476, or 5,496.

---

Fa = 870,17☐ and Fa is divisible by 5.

Fa could be 870,170 or 870,175

---

So = 1,3☐1 and So is divisible by 3.

So could be 1,311, 1,341, or 1,371.

---

First clue

Mip is a secret whole number smaller than 50.

Mip ⊓ 18 = 3

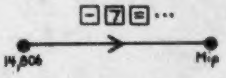
Mip could be 3, 15, 21, 33, or 39.

Second clue

Mip =, 3

Mip could be 3, 15, or 39.

Third clue

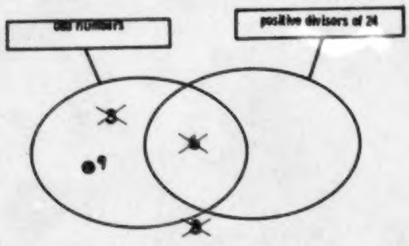


Mip is 15.

---

| The red tag is one of these: | The blue tag is one of these: |
|---|---|
| multiples of 4 | multiples of 4 |
| odd numbers | odd numbers |
| smaller than 10 | smaller than 10 |
| positive divisors of 12 | positive divisors of 12 |
| positive divisors of 18 | positive divisors of 18 |
| positive divisors of 24 | positive divisors of 24 |

Label the strings.



---

Mali is a secret number.

First clue

Mali is divisible by 4.

Mali = 201,620,3☐6

Mali's name can be completed by putting exactly one of these digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in the box.

Which digits could go in the box? 1, 3, 5, 7, or 9.

Second clue

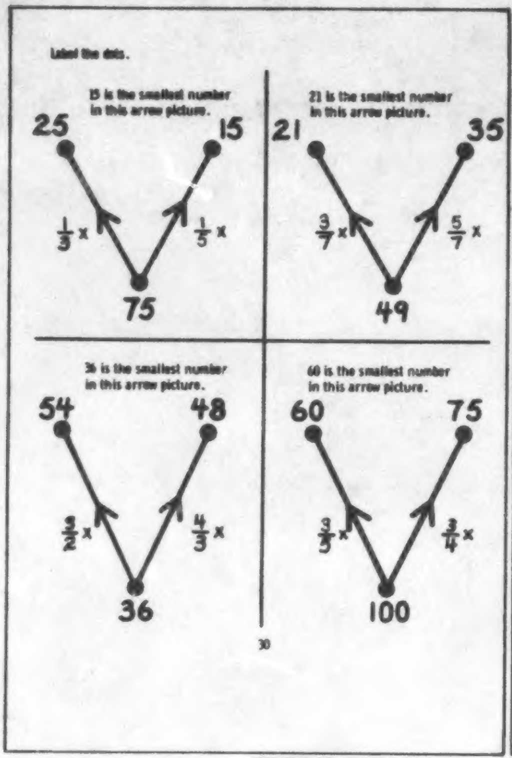
Mali is divisible by 8.

Mali could be 201,620,336 or 201,620,376.

Third clue

Mali is not divisible by 9.

Mali is 201,620,336.



Label the dots.

15 is the smallest number in this arrow picture.

25
15
$\frac{1}{3}$ x
$\frac{1}{5}$ x
75

21 is the smallest number in this arrow picture.

21
35
$\frac{3}{7}$ x
$\frac{5}{7}$ x
49

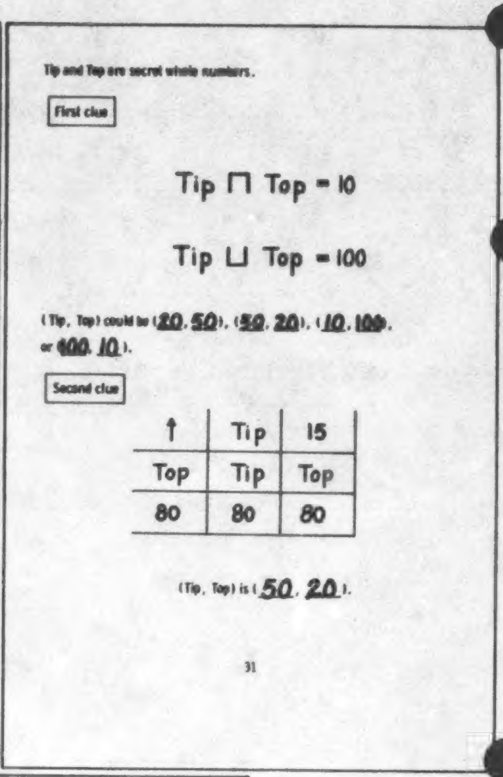
36 is the smallest number in this arrow picture.

54
48
$\frac{3}{2}$ x
$\frac{4}{3}$ x
36

60 is the smallest number in this arrow picture.

60
75
$\frac{3}{5}$ x
$\frac{3}{4}$ x
100

30



Tip and Top are secret whole numbers.

First clue

$$\text{Tip} \sqcap \text{Top} = 10$$

$$\text{Tip} \sqcup \text{Top} = 100$$

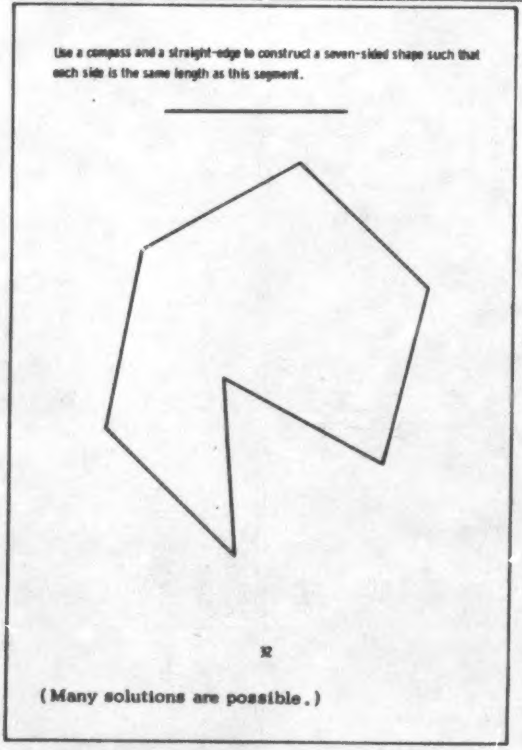
(Tip, Top) could be (20, 50), (50, 20), (10, 100), or (100, 10).

Second clue

| ↑ | Tip | 15 |
|---|---|---|
| Top | Tip | Top |
| 80 | 80 | 80 |

(Tip, Top) is (50, 20).

31



Use a compass and a straight-edge to construct a seven-sided shape such that each side is the same length as this segment.

32

(Many solutions are possible.)